

# Identification and Estimation of Causal Peer Effects Using Double Negative Controls for Unmeasured Network Confounding

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## Abstract

Scientists have been interested in estimating causal peer effects to understand how people's behaviors are affected by their network peers. However, it is well known that identification and estimation of causal peer effects are challenging in observational studies for two reasons. The first is the identification challenge due to unmeasured network confounding, for example, homophily bias and contextual confounding. The second issue is network dependence of observations, which one must take into account for valid statistical inference. Negative control variables, also known as placebo variables, have been widely used in observational studies including peer effect analysis over networks, although they have been used primarily for bias detection. In this article, we establish a formal framework which leverages a pair of negative control outcome and exposure variables (double negative controls) to nonparametrically identify causal peer effects in the presence of unmeasured network confounding. We then propose a generalized method of moments estimator for causal peer effects, and establish its consistency and asymptotic normality under an assumption about  $\psi$ -network dependence. Finally, we provide a network heteroskedasticity and autocorrelation consistent variance estimator. Our methods are illustrated with an application to peer effects in education.

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# 1 Introduction

Social and biomedical scientists have long been interested in how people’s behaviors are affected by peer behaviors. For example, scholars have studied peer effects on voting behaviors (Sinclair, 2012; Jones *et al.*, 2017), educational outcomes (Epple and Romano, 2011; Sacerdote, 2011), criminal behaviors (Glaeser *et al.*, 1996), and job opportunities (Granovetter, 1973). Epidemiologists and researchers in public health have studied related concepts such as “contagion” effects of infectious disease (Halloran and Struchiner, 1995; Morozova *et al.*, 2018) and health behaviors (Christakis and Fowler, 2013).

Despite its importance, identification and estimation of causal peer effects have been challenging for two reasons. The first issue is that it is often difficult to identify causal peer effects in observational studies due to unmeasured network confounding, such as *homophily bias* and *contextual confounding* (Manski, 1993; VanderWeele and An, 2013; Ogburn, 2018). Homophily bias arises when people become connected due to unobserved characteristics. Contextual confounding exists when peers share some unobserved contextual factors. Highlighting concerns about these potential biases, influential papers across disciplines have criticized prior peer effect analyses from observational studies (e.g., Lyons, 2011; Angrist, 2014). Shalizi and Thomas (2011) argue that it is nearly impossible to credibly estimate causal peer effects in observational studies using direct confounding adjustment methods (e.g. regression based adjustment) due to pervasive concerns about unmeasured network confounding.

In addition to unmeasured network confounding, another important challenge is that one needs to account for network dependence of observations in order to obtain valid statistical inference. When such network dependence is ignored, standard errors may be underestimated, and confidence intervals may be anti-conservative (Lee and Ogburn, 2020). While recent studies have allowed for some extent of network dependence across units in observational causal inference (van der Laan, 2014; Ogburn *et al.*, 2017; Forastiere *et al.*, 2020; Ogburn *et al.*, 2020; Tchetgen Tchetgen *et al.*, 2020b; Leung, 2021), they have largely relied on an assumption of no uncontrolled network confounding.

In this paper, we propose to resolve these challenges by using a pair of negative control outcome and exposure variables (*double negative controls*). A negative control outcome (also known as a placebo outcome) is an outcome variable that is known not to be causally affected by the treatment of interest. Likewise, a negative control exposure (also known as a placebo treatment) is a treatment variable that does not causally affect the outcome of interest (Lipsitch *et al.*, 2010). There is a long-standing tradition in biomedical and social sciences of using negative controls to detect unmeasured confounding. A non-null effect of the treatment on the

negative control outcome or a non-null effect of the negative control exposure on the outcome of interest amounts to compelling evidence of unmeasured confounding. In the literature of causal peer effects, Egami (2018) exploits a negative control outcome, and Liu and Tchetgen Tchetgen (2020) use a negative control exposure to address unmeasured network confounding. While they require relatively weak assumptions to detect unmeasured network confounding, both works require much stronger assumptions for identification of causal peer effects as they each use only one type of negative control variable but not both. Recently, a series of papers (Kuroki and Pearl, 2014; Miao *et al.*, 2018a,b; Deaner, 2018; Shi *et al.*, 2020; Tchetgen Tchetgen *et al.*, 2020a; Kallus *et al.*, 2021) propose to use double negative controls for identification of causal effects, but they have focused on i.i.d or panel data settings, and to date none of these papers has considered network data.

Our contribution is to propose a general framework for using double negative controls for identification and estimation of causal peer effects in the presence of uncontrolled network confounding, while taking into account network dependence. We first derive nonparametric identification of causal peer effects in the presence of unmeasured network confounding by exploiting double negative controls that are associated with unmeasured confounders. In particular, we incorporate double negative control variables via a network outcome confounding bridge function, a network version of the outcome confounding bridge function studied in Miao *et al.* (2018a,b); Shi *et al.* (2020); Tchetgen Tchetgen *et al.* (2020a). We discuss general approaches for selecting negative controls from network data in practice. We then propose a generalized method of moments (GMM) estimator (Hansen, 1982) for the causal peer effect, and we establish consistency and asymptotic normality of the resulting estimator under correct specification of the network confounding bridge function, and an assumption about  $\psi$ -network dependence (Kojevnikov *et al.*, 2020), which expresses the degree of stochastic dependence between variables in terms of network distance. This assumption of  $\psi$ -network dependence restricts the speed by which network dependence decays as network distance increases, and the speed by which the density of the network changes as sample size increases. Finally, we provide a network heteroskedasticity and autocorrelation consistent (network HAC) variance estimator, with which researchers can construct asymptotic confidence intervals of causal peer effects.

The paper is organized as follows. In Section 2, we consider dyadic data to focus on the use of double negative controls for identification of causal peer effects. A corresponding framework for estimation and inference via GMMs is relatively straightforward in this setting assuming a sample of independent and identically distributed dyads is available. In Section 3, we examine a more general setting where one observes data from a single network. We study identification

as well as estimation and inference by accounting for  $\psi$ -network dependence. In Section 4, we assess the finite sample performance of our proposed estimators via extensive simulations. Also, we illustrate our methods by applying them to the Add Health network data to infer the extent of causal peer effects in education in Section 5. We extend our results in Section 6 to settings where researchers are interested in causal peer effects from higher-order peers (i.e., those not directly connected to a given focal unit). Section 7 concludes the paper.

## Related Literature

This article builds on a growing literature on causal peer effects. Various methods have been proposed to address concerns about uncontrolled network confounding in observational studies. A popular approach for identification of causal peer effects is the so-called instrumental variable method. Bramoullé *et al.* (2009) use instrumental variables to deal with simultaneity in the linear-in-mean models (Manski, 1993; Goldsmith-Pinkham and Imbens, 2013). O’Malley *et al.* (2014) propose to use genes as instruments to study causal peer effects of body mass index among friends. In addition to the well known exclusion restriction, both methods assume (conditional) exogeneity of instrumental variables. However, this assumption may be untenable in a wide range of applications because it is violated as long as instrumental variables are associated with unmeasured variables at the source of homophily or contextual confounding. In contrast, our methods allow for and in fact leverage such association between negative controls and unobserved confounders. McFowland III and Shalizi (2021) propose a consistent estimator of causal peer effects, which adjusts for estimated latent homophilous attributes in settings where the data generating process is linear and the network grows according to either a stochastic block model or a continuous space model. In contrast, we establish nonparametric identification of causal peer effects by using double negative controls, and our results accommodate both latent homophily and contextual confounding. Also, our asymptotic results make an alternative assumption of  $\psi$ -network dependence (Kojevnikov *et al.*, 2020) instead of assuming specific network models, and we allow for asymptotic normality and construction of asymptotic confidence intervals in addition to consistent estimation of causal peer effects.

There is also a literature focusing on a different research goal, such as testing (e.g., Anagnostopoulos *et al.*, 2008; VanderWeele *et al.*, 2012), partial identification (Ver Steeg and Galstyan, 2010, 2013), and sensitivity analysis (VanderWeele, 2011), rather than (point) identification of causal peer effects. Finally, our methods addressing unmeasured network confounding in observational studies are complementary to approaches based on randomized experiments or natural experiments (Sacerdote, 2001; Duflo *et al.*, 2011; Rogowski and Sinclair, 2012; Taylor and Eckles, 2017; Basse *et al.*, 2019; Li *et al.*, 2019).

## 2 Double Negative Controls for Dyadic Data

In this section, in order to ground ideas, we focus on identification and estimation of causal peer effects from dyadic data. In general, even in this simple setting, causal peer effects are not identified based on standard covariate adjustment in the presence of unmeasured network confounding, such as latent homophily and contextual confounding. We propose an alternative approach based on negative controls. We consider a more general network setting in Section 3.

### 2.1 Notation and Definitions

We consider data on dyads, i.e., pairs of two individuals. For each dyad  $i$ , let  $S_i = 1$  when the two units are connected and  $S_i = 0$  when the two units have no tie between them. For example, in dyadic data based on a students' friendship survey,  $S_i = 1$  encodes the two students being friends with each other and  $S_i = 0$  otherwise.

Suppose one has observed  $n$  independent and identically distributed samples of connected dyads ( $S_i = 1$ ) where each dyad is labeled  $i \in \{1, \dots, n\}$ . For each unit within dyads, we observe focal behavior  $Y$  at two time points, baseline and a single follow-up. Define  $Y_{kt}$  to be the focal behavior of unit  $k \in \{1, 2\}$  at time  $t \in \{1, 2\}$  where  $t = 1, 2$  denotes baseline and follow-up, respectively. Without loss of generality, define  $k = 1$  to be the *ego* — a unit on whom we estimate a causal peer effect — and define  $k = 2$  to be the *peer* — a unit whose effect on the ego we estimate.

The outcome of interest is ego's behavior at follow-up  $Y_{12}$ . The treatment variable of interest is peer's behavior at the baseline  $Y_{21}$ . Using the potential outcomes framework (Neyman, 1923; Rubin, 1974; Robins, 1986), define  $Y_{12}(y_{21})$  to be the potential outcome had possibly contrary to fact, the treatment variable been set to  $Y_{21} = y_{21}$ , which we assume to exist and to be well defined. Throughout the paper, we make the standard consistency assumption linking observed and potential outcomes:

$$Y_{12} = Y_{12}(Y_{21}), \quad a.s. \tag{1}$$

Our goal is to estimate the Average Causal Peer Effect (ACPE), defined as

$$\tau(y_{21}, y'_{21}) := \mathbb{E}\{Y_{12}(y_{21}) - Y_{12}(y'_{21}) \mid S = 1\}$$

where  $y_{21}, y'_{21} \in \mathcal{Y}_{21}$  where  $\mathcal{Y}_{21}$  is the support of  $Y_{21}$ . We condition on  $S = 1$  because we only consider connected dyads.

### 2.2 Identification Challenge

Covariate adjustment based on conditional ignorability is the most common approach to identification of causal effects in observational studies (Rosenbaum and Rubin, 1983; Robins, 1986).

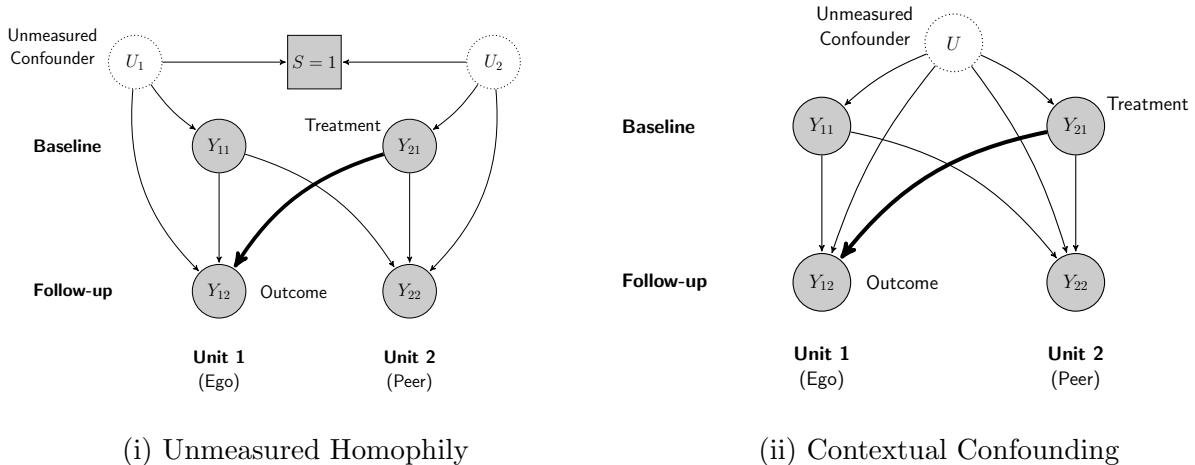


Figure 1: A DAG for dyadic data in the presence of unmeasured network confounding. *Note:* The thick arrow from  $Y_{21}$  to  $Y_{12}$  indicates the causal peer effect of interest. We use shaded (dotted) nodes to denote observed (unobserved) variables. For simplicity, we have suppressed observed covariates  $\mathbf{X}$ . In Figure 1.(i), the square box around  $S = 1$  represents that we observe dyads conditional on  $S = 1$ .

In causal peer analysis, conditional ignorability entails assuming

$$Y_{12}(y_{21}) \perp\!\!\!\perp Y_{21} \mid Y_{11}, \mathbf{X}, S = 1 \quad (2)$$

where  $\mathbf{X}$  represent observed pre-treatment covariates for the dyad. However, such approach is, in general, not plausible when estimating the ACPE due to unmeasured homophily (Shalizi and Thomas, 2011) and contextual confounding (VanderWeele and An, 2013). Homophily bias arises when people become connected due to unobserved characteristics. Contextual confounding exists when peers share some unobserved contextual factors. In this paper, we use the term *unmeasured network confounding* to refer to confounding that violates conditional ignorability in causal peer analysis, and thus it contains unmeasured homophily and contextual confounding as special cases.

Figure 1.(i) represents a causal directed acyclic graph (DAG) illustrating latent homophily. We use  $U_k$  with  $k \in \{1, 2\}$  to represent unobserved characteristics that affect a tie relationship  $S$  as well as focal behavior  $Y$ . Because the tie relationship  $S$  is affected by  $U_1$  and  $U_2$ , variable  $S$  is a collider in the terminology of graph theory (Pearl, 2009). To estimate the ACPE, we condition on  $S = 1$ , and therefore, there is an unblocked backdoor path  $Y_{21} \leftarrow U_2 \rightarrow \boxed{S = 1} \leftarrow U_1 \rightarrow Y_{12}$  where  $U_1$  and  $U_2$  are unobserved, and the square box around  $S = 1$  means that we observe dyads conditional on  $S = 1$  (Shalizi and Thomas, 2011). Thus, even in the simple setting of dyadic data, identification of the ACPE is not possible without additional assumptions.

Figure 1.(ii) represents a causal DAG illustrating contextual confounding. Here, we use  $U$

to represent an unmeasured shared context that affects both the ego and peer. Due to the unblocked back-door path  $Y_{21} \leftarrow U \rightarrow Y_{12}$ , conditional ignorability is violated.

### 2.3 Identification with Double Negative Controls

In this section, we consider an alternative approach for identification and estimation by exploiting auxiliary variables called negative controls. In particular, we will use negative control outcome (NCO) and negative control exposure (NCE) variables, which we define below.

We first make the latent ignorability assumption, which states that conditional ignorability holds if we could measure all factors at the source of network confounding.

**Assumption 1.1 (Latent Ignorability)** *For all  $y_{21} \in \mathcal{Y}_{21}$ ,*

$$Y_{12}(y_{21}) \perp\!\!\!\perp Y_{21} \mid U, \mathbf{X}, S = 1.$$

Assumption 1.1 states that  $U$ ,  $\mathbf{X}$ , and  $S$  suffice to account for confounding of the relationship between  $Y_{21}$  and  $Y_{12}(y_{21})$ , whereas  $\mathbf{X}$  and  $S$  alone may not. This assumption is often plausible as there is no direct restriction on the nature of the latent characteristic  $U$ . However, this assumption alone is not sufficient for identification given that we do not observe the latent characteristic  $U$ .

The key to the proposed approach is to suppose that one can measure two auxiliary variables, negative control outcome  $W$  and negative control exposure  $Z$  that satisfy the following conditions.

**Assumption 1.2 (Negative Controls)**

1. Negative Control Outcome (NCO):

$$W \perp\!\!\!\perp Y_{21} \mid U, \mathbf{X}, S = 1 \tag{3}$$

2. Negative Control Exposure (NCE):

$$Z \perp\!\!\!\perp Y_{12} \mid Y_{21}, U, \mathbf{X}, S = 1 \quad \text{and} \quad Z \perp\!\!\!\perp W \mid Y_{21}, U, \mathbf{X}, S = 1. \tag{4}$$

Assumption 1.2.1 states that  $W$  is an auxiliary variable that is conditionally independent of the treatment  $Y_{21}$  given the latent confounder  $U$ , observed pre-treatment covariates  $\mathbf{X}$ , and the dyadic type  $S$ . Assumption 1.2.2 means that  $Z$  is an auxiliary variable that is conditionally independent of the outcome  $Y_{12}$  and NCO  $W$  given the treatment  $Y_{21}$ , the latent confounder  $U$ , observed pre-treatment covariates  $\mathbf{X}$ .

In practice, plausible candidates for negative controls are auxiliary variables that (a) do not affect network relationships and (b) do not affect variables of other units. Figure 2 represents

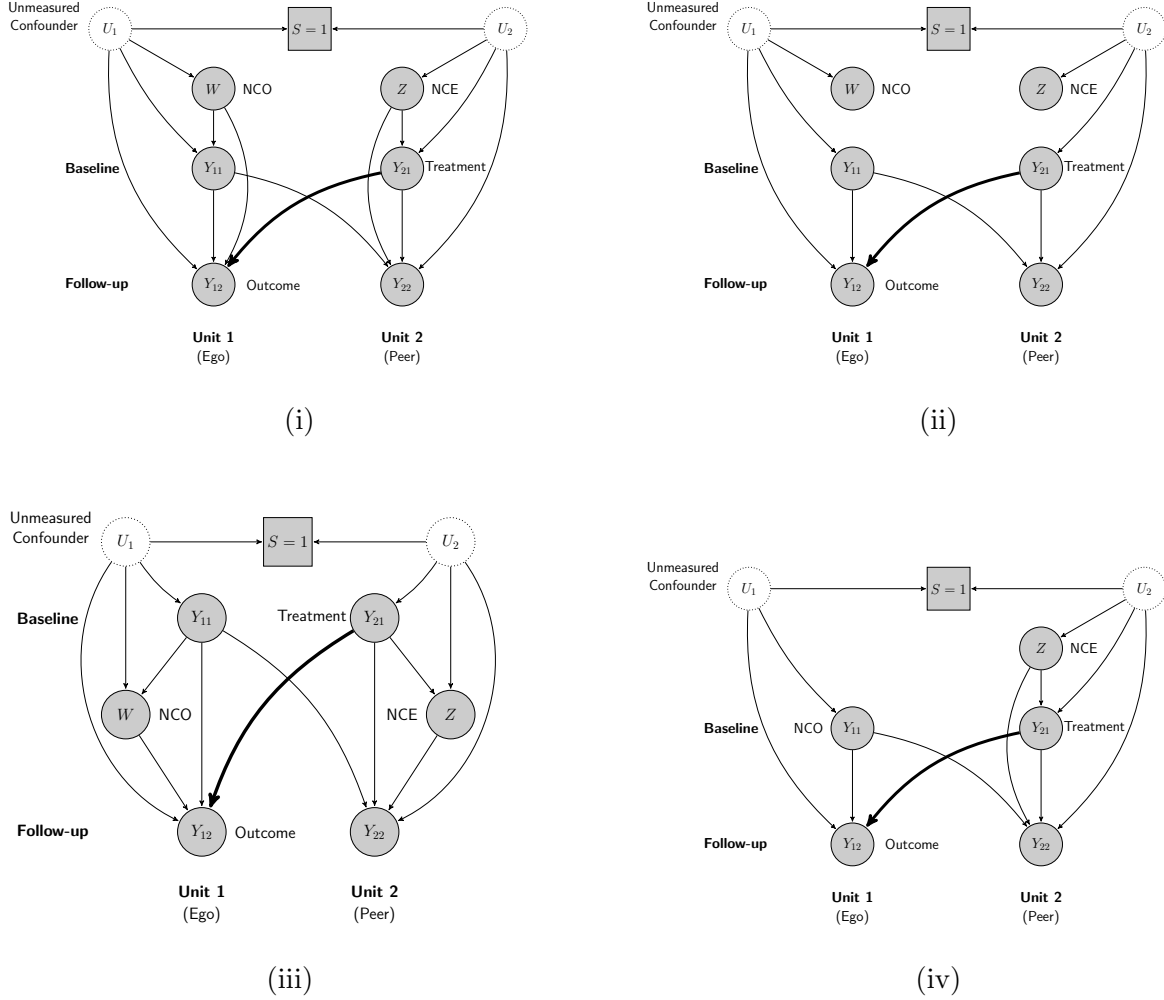


Figure 2: Examples of causal DAGs with Double Negative Controls. *Note:* Auxiliary variables  $W$  and  $Z$  are added to Figure 1.(i). The causal relationships between focal behaviors  $Y_{kt}$ , unmeasured confounder  $U_k$ , and the tie relationship  $S$  are the same as those in Figure 1.(i).

examples of causal DAGs that extend Figure 1.(i) and encode Assumption 1.2. In Figures 2.(i)–(iii),  $W$  and  $Z$  are auxiliary variables that do not affect the dyadic relationship  $S$  or do not affect variables of the other unit. A variety of relationships between the focal behaviors and negative controls can be accommodated. In Figure 2.(i),  $W$  and  $Z$  are pre-treatment variables affecting the focal behaviors, while in Figure 2.(ii),  $W$  and  $Z$  have no causal relationship with the focal behaviors. In Figure 2.(iii),  $W$  and  $Z$  are intermediate variables between focal behavior at baseline  $Y_{k1}$  and focal behavior at follow-up  $Y_{k2}$ . Figure 2.(iv) shows that focal behavior at baseline  $Y_{11}$  can also serve as NCO in the scenario represented by Figure 2.(i). Indeed, in all Figures 2.(i)–(iii),  $Y_{11}$  may also serve as NCO.

**Example (Negative Controls).** In the context of Add Health data, Cohen-Cole and Fletcher (2008) estimated peer effects on three health outcomes — acne, height, and headaches — known



ex ante not to exhibit peer effects. The purpose of Cohen-Cole and Fletcher (2008)’s analysis was to investigate the validity of popular statistical approaches that assume the absence of unmeasured homophily and contextual confounding (e.g., Christakis and Fowler, 2007). By finding implausible peer effects on the three health outcomes mentioned above, Cohen-Cole and Fletcher (2008) warn that unmeasured homophily and contextual confounding may be operating in studies of peer effects based on Add Health data.

In the proposed double negative control approach, the three health outcomes can be used not only for detecting confounding but also as negative controls to potentially correct for such confounding. For example, whether a student has headaches is unlikely (a) to causally affect whether students are friends to each other and (b) to causally affect peers’ headaches. If these conditions are plausible in applied contexts, an ego’s headache and a peer’s headache can be used as the NCO and NCE, respectively.  $\square$

Upon selecting valid NCO and NCE variables, one approach supposes that there exists an outcome confounding bridge (Miao *et al.*, 2018a) that relates the confounders’ effects on negative control outcome  $W$  to the confounders’ effects on outcome of primary interest  $Y_{12}$ .

**Assumption 1.3 (Outcome Confounding Bridge)** *There exists a function  $h(W, Y_{21}, \mathbf{X})$  such that for all  $y_{21} \in \mathcal{Y}_{21}$ ,*

$$\mathbb{E}(Y_{12} \mid Y_{21} = y_{21}, U, \mathbf{X}, S = 1) = \mathbb{E}\{h(W, y_{21}, \mathbf{X}) \mid Y_{21} = y_{21}, U, \mathbf{X}, S = 1\}. \quad (5)$$

Assumption 1.3 states that the confounding effect of  $U$  on outcome  $Y_{12}$  is equal to the confounding effect of  $U$  on  $h(W, y_{21}, \mathbf{X})$ , a transformation of  $W$ . One simple yet important implication of this assumption is that  $W$  should be associated with  $U$  conditional on the treatment, observed covariates, and the dyadic relationship.

This assumption formally connects the confounding effect on the outcome  $Y_{12}$  and the confounding effect on the negative control outcome  $W$ . Instead of assuming complete knowledge of its relationship, the proposed double negative control approach will use negative control exposure  $Z$  to identify it, as described below in Theorem 1.

Formally, equation (5) is a Fredholm integral equation of the first kind (Kress, 1989; Carrasco *et al.*, 2007). Existence of a solution to this equation can be established under regularity conditions regarding the NCO relevance and a certain singular value decomposition of the operator defining the integral equation. These conditions are somewhat technical and so we reserve their details to Section A.1 in the supplementary material, where we provide further discussion and a proof of the following lemma.

**Lemma 1** *Under Assumptions 5 and 6 (defined in Section A.1), there exists a function  $h(W, Y_{21}, \mathbf{X})$  such that for all  $y_{21} \in \mathcal{Y}_{21}$ , equation (5) holds.*

**Example (Linear Confounding Bridge).** While the bridge function  $h$  can take any functional form, we illustrate the assumption with a linear confounding bridge. Suppose that  $\mathbb{E}(Y_{12} | Y_{21}, U, \mathbf{X}, S = 1) = (1, Y_{21}, U, \mathbf{X})^\top \beta$  and that  $\mathbb{E}(W | U, \mathbf{X}, S = 1)$  is linear in  $U$  and  $\mathbf{X}$ , then equation (5) holds with  $h(W, Y_{21}, \mathbf{X}; \gamma) = (1, W, Y_{21}, \mathbf{X})^\top \gamma$ , with an appropriate value of  $\gamma$ . We do not need to assume the value of  $\gamma$ . Rather, as shown below, we can use an appropriate choice of negative control exposure  $Z$  to identify and estimate  $\gamma$ .  $\square$

**Example (Categorical Variables).** Suppose  $W, U, Z, Y_{21}$  are all categorical variables. We define possible values for  $W$  and  $U$  as  $w_i$  and  $u_j$  for  $i = 1, \dots, |W|, j = 1, \dots, |U|$  where  $|\cdot|$  denotes the cardinality of a categorical variable.  $|Y_{21}|$  and  $|Z|$  are also similarly defined. Define also  $\Pr(W | y_{21}, U, \mathbf{x}, S = 1)$  to be a  $|W| \times |U|$  matrix with  $\Pr(W | y_{21}, U, \mathbf{x}, S = 1)_{ij} = \Pr(W = w_i | y_{21}, U = u_j, \mathbf{x}, S = 1)$ . As long as  $\Pr(W | y_{21}, U, \mathbf{x}, S = 1)$  have full column rank (which implies that  $|W| \geq |U|$ ), the confounding bridge  $h(W, y_{21}, \mathbf{x})$  exists as a solution to  $\mathbb{E}(Y_{12} | y_{21}, U, \mathbf{x}, S = 1) = h(W, y_{21}, \mathbf{x}) \Pr(W | y_{21}, U, \mathbf{x}, S = 1)$  where  $h(W, y_{21}, \mathbf{x})$  is a  $1 \times |W|$  vector. When the number of categories for  $U$  is equal to that of  $W$ , the confounding bridge function is unique and given by  $h(W, y_{21}, \mathbf{x}) = \mathbb{E}(Y_{12} | y_{21}, U, \mathbf{x}, S = 1) \Pr(W | y_{21}, U, \mathbf{x}, S = 1)^{-1}$ .  $\square$

Finally, we make the following assumption about negative control relevance.

**Assumption 1.4 (Negative Control Relevance)** *For any square integrable function  $f$  and any  $y_{21}$  and  $\mathbf{x}$ , if  $\mathbb{E}\{f(W) | Z = z, Y_{21} = y_{21}, \mathbf{X} = \mathbf{x}, S = 1\} = 0$  for almost all  $z$ , then  $f(W) = 0$  almost surely.*

This assumption states that  $Z$  is sufficiently informative about  $W$ , which is essential to ensure identification of the outcome bridge function  $h$ . This condition is formally known as a completeness condition, a well known technical condition central to the study of sufficiency in statistical inference (Casella and Berger, 2001). Many commonly-used parametric and semi-parametric models, such as semi-parametric exponential family (Newey and Powell, 2003) and semiparametric location-scale family (Hu and Shiu, 2018), satisfy the completeness condition.

The completeness condition has been widely used to achieve identification in non-parametric instrumental variable models (e.g., D'Haultfoeuille, 2011; Darolles *et al.*, 2011). In the non-parametric instrumental variable literature, completeness is an instrumental variable relevance condition, which generalizes the rank condition of linear instrumental variable models (Newey and Powell, 2003). Thus Assumption 1.4 essentially means that  $Z$  is a relevant variable for  $W$  conditional on  $(Y_{21}, \mathbf{X}, S)$ .

**Examples.** To gain further intuition, we consider implications of the completeness condition.

(Categorical Variables). Consider a special case of categorical NCO and NCE. In this case, Assumption 1.4 requires that the number of levels in NCE must be at least as large as the number of levels in NCO.

(Continuous Variables). When both NCO and NCE are continuous variables, Assumption 1.4 requires that the number of NCEs must be at least as large as the number of NCOs.

(Parametric or Semiparametric Confounding Bridge). While Assumption 1.4 is important for accommodating a nonparametric confounding bridge function, we can relax the completeness condition when a bridge function belongs to a parametric or semiparametric model  $h(W, Y_{21}, \mathbf{X}; \gamma)$  indexed by a finite or infinite dimensional parameter  $\gamma$ . Under such a model, the completeness condition only requires that, for all  $y_{21}$  and  $\mathbf{x}$ ,  $\mathbb{E}\{h(W, y_{21}, \mathbf{x}; \gamma) - h(W, y_{21}, \mathbf{x}; \gamma') \mid Z, \mathbf{X} = \mathbf{x}, Y_{21} = y_{21}, S = 1\} \neq 0$  with a positive probability for any  $\gamma \neq \gamma'$  (see Miao *et al.*, 2018a, for further details).  $\square$

**Remark.** We note that alternative completeness conditions may also be sufficient for identification of the ACPE. While completeness condition (Assumption 1.4) is analogous to the one used in Miao *et al.* (2018a), alternative completeness conditions have also been considered in related studies (e.g., Deaner, 2018; Miao *et al.*, 2018b; Shi *et al.*, 2020; Kallus *et al.*, 2021), all in the more tractable i.i.d. or panel data settings and not in network settings. In addition to Theorem 1 given below, for the sake of completeness, we also establish nonparametric identification of the ACPE under alternative identifying conditions in Section A.6 of the supplementary material. We further discuss completeness condition in Section A.2 of the supplementary material. Interested readers can also see Chen *et al.* (2014) and Andrews (2017), and references therein for an overview of the role of completeness in nonparametric causal inference.  $\square$

The following theorem demonstrates nonparametric identification of the ACPE under the stated assumptions.

**Theorem 1** *Under Assumptions 1.1-1.4, the confounding bridge function is identified as the unique solution to the following equation.*

$$\mathbb{E}(Y_{12} \mid Z, Y_{21}, \mathbf{X}, S = 1) = \mathbb{E}\{h(W, Y_{21}, \mathbf{X}) \mid Z, Y_{21}, \mathbf{X}, S = 1\}.$$

Using the identified confounding bridge function  $h(W, Y_{21}, \mathbf{X})$ , the ACPE is identified by

$$\tau(y_{21}, y'_{21}) = \mathbb{E}\{h(W, y_{21}, \mathbf{X}) - h(W, y'_{21}, \mathbf{X}) \mid S = 1\}.$$

Importantly, under Assumptions 1.1-1.4, we have identified the ACPE without imposing any parametric restriction on the confounding bridge or negative controls. We provide a proof in Section A.5 of the supplementary material.

**Example (Identification under Linear Confounding Bridge).** While Theorem 1 establishes nonparametric identification of the ACPE, here we consider a simple case with a binary treatment  $Y_{21}$  and a binary NCE  $Z$  without any pre-treatment covariates, which admits a closed-form solution. Suppose the confounding bridge function is linear:  $h(W, Y_{21}; \gamma) = \gamma_0 + \gamma_1 W + \gamma_2 Y_{21}$ . Then, the ACPE is identified as

$$\tau(1, 0) = \mathbb{E}(\text{OD}_{Y_{12}Y_{21}|Z}) - \mathbb{E}(\text{OD}_{WY_{21}|Z}) \times \frac{\mathbb{E}(\text{OD}_{Y_{12}Z|Y_{21}})}{\mathbb{E}(\text{OD}_{WZ|Y_{21}})}$$

where  $\text{OD}_{V_1V_2|V_3} = \mathbb{E}(V_1 | V_2 = 1, V_3, S = 1) - \mathbb{E}(V_1 | V_2 = 0, V_3, S = 1)$ .

The first term  $\mathbb{E}(\text{OD}_{Y_{12}Y_{21}|Z})$  corresponds to a biased estimator of the ACPE; a regression of outcome  $Y_{12}$  on treatment  $Y_{21}$  conditional on  $Z$ , which is equal to the ACPE only in the absence of unmeasured confounder. The second term  $\mathbb{E}(\text{OD}_{WY_{21}|Z})$  corresponds to an estimator of the confounding effect on  $W$ , which should be zero in the absence of unmeasured confounding. This captures the amount of confounding to be corrected. Finally, the third term  $\mathbb{E}(\text{OD}_{Y_{12}Z|Y_{21}})/\mathbb{E}(\text{OD}_{WZ|Y_{21}})$  represents a ratio of the confounding effects on the outcome  $Y_{12}$  and on the negative control outcome  $W$ , which is estimated by using NCE  $Z$ . It is clear here that the negative control relevance (Assumption 1.4) is essential to guarantee  $\mathbb{E}(\text{OD}_{WZ|Y_{21}}) \neq 0$ . Intuitively, the double negative control approach subtracts an estimated bias (the second term) scaled by the differential confounding effects on the outcome and on the NCO (the third term) from the original biased estimator (the first term).

This explicit form contains two important special cases: (1) conditional ignorability (i.e., no unmeasured network confounding), and (2) the well-known difference-in-differences (DID) design (Angrist and Pischke, 2008). First, when conditional ignorability holds and there exists no unmeasured network confounding,  $\mathbb{E}(\text{OD}_{WY_{21}|Z}) = 0$  and  $\tau(1, 0) = \mathbb{E}(\text{OD}_{Y_{12}Y_{21}|Z})$ , which reduces to the standard identification formula under conditional ignorability (Rosenbaum and Rubin, 1983; Robins, 1986). Second, the formula reduces to DID when we assume the confounding effect on the outcome is equal to the confounding effect on the NCO. Sofer *et al.* (2016) show that the DID uses a pre-treatment outcome ( $Y_{11}$  in our setting) as  $W$  and assumes the entire bridge function is known, i.e., not only a functional form but also the value of coefficients  $\gamma$ , without using any NCE. In particular, the widely-used assumption of parallel trends assumes that the confounding effects on the outcome  $Y_{12}$  and on pre-treatment outcome  $Y_{11}$  (used as the NCO) are the same, i.e., assuming the third term equal to one.

In contrast, the double negative control approach can use any valid  $W$  (including pre-treatment outcome  $Y_{11}$  as a special case; see Figure 2.(iv)). Most importantly, we use NCE  $Z$  to estimate the confounding bridge function — the differential confounding effects on the outcome  $Y_{12}$  and on the negative control outcome  $W$  — as the ratio  $\mathbb{E}(\text{OD}_{Y_{12}Z|Y_{21}})/\mathbb{E}(\text{OD}_{WZ|Y_{21}})$  in the third term. It is important to emphasize that while we use this closed-form solution in the linear confounding bridge case to illustrate the intuition behind the double negative control approach, our identification results do not impose any parametric restriction on the confounding bridge or negative controls.  $\square$

## 2.4 Estimation and Inference

We now propose a strategy for estimation and inference of the ACPE. Because we observe  $n$  independent and identically distributed samples of dyads, we observe independent and identically distributed samples on  $(Y_{12}, Y_{21}, W, Z, \mathbf{X})$  given  $S = 1$  where  $Y_{12}, Y_{21}, W, Z, \mathbf{X}$  are the outcome of interest, treatment, NCO, NCE, and observed pre-treatment covariates, respectively.

Suppose that an analyst has specified a parametric or semiparametric model for the confounding bridge  $h(W, Y_{21}, \mathbf{X}; \gamma)$  with parameter  $\gamma$ . Then, based on Theorem 1, we can estimate  $\gamma$  by solving the following empirical moment equations.

$$\frac{1}{n} \sum_{i=1}^n \{Y_{i12} - h(W_i, Y_{i21}, \mathbf{X}_i; \gamma)\} \times \eta(Z_i, Y_{i21}, \mathbf{X}_i) = 0,$$

where  $\eta$  is a user-specified vector function with dimension equal to that of  $\gamma$ . For example, if a linear confounding bridge function is used, i.e.,  $h(W, Y_{21}, \mathbf{X}; \gamma) = (1, W, Y_{21}, \mathbf{X})^\top \gamma$ , we can use  $\eta(Z, Y_{21}, \mathbf{X}) = (1, Z, Y_{21}, \mathbf{X})^\top$ .

Once the bridge function  $h$  is estimated, we can estimate the ACPE by

$$\frac{1}{n} \sum_{i=1}^n \{h(W_i, y_{21}, \mathbf{X}_i; \hat{\gamma}) - h(W_i, y'_{21}, \mathbf{X}_i; \hat{\gamma})\}.$$

To appropriately account for uncertainty of the estimated bridge function and for the possibility that dimension of  $\eta$  might be larger than that of  $\gamma$ , we combine the two moments into generalized method of moments (GMM) with parameter  $\theta = (\tau, \gamma)$  (Hansen, 1982). We define a moment for dyad  $i$  to be

$$m(Y_{i12}, Y_{i21}, W_i, Z_i, \mathbf{X}_i; \theta) = \left\{ \begin{array}{l} \tau - \{h(W_i, y_{21}, \mathbf{X}_i; \gamma) - h(W_i, y'_{21}, \mathbf{X}_i; \gamma)\} \\ \{Y_{i12} - h(W_i, Y_{i21}, \mathbf{X}_i; \gamma)\} \times \eta(Z_i, Y_{i21}, \mathbf{X}_i) \end{array} \right\}.$$

Then, the GMM estimator is

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \bar{m}(\theta)^\top \Omega \bar{m}(\theta) \tag{6}$$

where  $\bar{m}(\theta) = \frac{1}{n} \sum_{i=1}^n m(Y_{i12}, Y_{i21}, W_i, Z_i, \mathbf{X}_i; \theta)$  and  $\Omega$  is a user-specified positive-definite weight matrix. Asymptotic properties described below hold for any positive-definite weight matrix  $\Omega$ .

The proposed double negative control (DNC) estimator  $\hat{\tau}(y_{21}, y'_{21})$  for  $\tau(y_{21}, y'_{21})$  is the first element of  $\hat{\theta}$  defined in equation (6). Because we consider i.i.d samples of dyads in this section, the moment  $m(Y_{i12}, Y_{i21}, W_i, Z_i, \mathbf{X}_i; \theta)$  is also i.i.d., and thus, under the standard regularity conditions for GMM (Hansen, 1982; Newey and McFadden, 1994), the DNC estimator is consistent:

$$\hat{\tau}(y_{21}, y'_{21}) \xrightarrow{p} \tau(y_{21}, y'_{21}),$$

and asymptotically normal:

$$\frac{\hat{\tau}(y_{21}, y'_{21}) - \tau(y_{21}, y'_{21})}{\sqrt{\sigma^2/n}} \xrightarrow{d} \text{Normal}(0, 1),$$

where  $\xrightarrow{p}$  denotes convergence in probability, and  $\xrightarrow{d}$  denotes convergence in distribution. Moreover, the asymptotic variance  $\sigma^2$  can be consistently estimated by  $\hat{\sigma}^2 = (\hat{\Gamma}\hat{\Lambda}\hat{\Gamma}^\top)_{11}$ , which is the (1, 1) th element of matrix  $\hat{\Gamma}\hat{\Lambda}\hat{\Gamma}^\top$ , and

$$\begin{aligned} \hat{\Lambda} &= \frac{1}{n} \sum_{i=1}^n m(Y_{i12}, Y_{i21}, W_i, Z_i, \mathbf{X}_i; \hat{\theta}) m(Y_{i12}, Y_{i21}, W_i, Z_i, \mathbf{X}_i; \hat{\theta})^\top, \\ \hat{\Gamma} &= (\hat{M}^\top \Omega \hat{M})^{-1} \hat{M}^\top \Omega, \quad \text{and} \quad \hat{M} = \frac{1}{n} \sum_{i=1}^n \frac{\partial}{\partial \theta} m(Y_{i12}, Y_{i21}, W_i, Z_i, \mathbf{X}_i; \hat{\theta}). \end{aligned}$$

Therefore, an asymptotically valid  $(1-\alpha)$  confidence interval for  $\tau(y_{21}, y'_{21})$  is given by  $[\hat{\tau}(y_{21}, y'_{21}) - \Phi(1-\alpha/2) \times \hat{\sigma}/\sqrt{n}, \hat{\tau}(y_{21}, y'_{21}) + \Phi(1-\alpha/2) \times \hat{\sigma}/\sqrt{n}]$  where  $\Phi(\cdot)$  denotes the quantile function for the standard normal distribution.

To minimize the asymptotic variance within the GMM class, we can use the two-step GMM to estimate the optimal  $\hat{\Omega}$ . In the first step, we choose an identity matrix as  $\Omega$  or some other positive-definite matrix, and compute preliminary GMM estimate  $\hat{\theta}_{(1)}$ . This estimator is consistent, but not efficient. In the second step, we compute  $\hat{\Lambda}$  based on  $\hat{\theta}_{(1)}$ , which is denoted by  $\hat{\Lambda}_{(1)}$ . Then, we can get the final estimate by solving equation (6) with  $\Omega = \hat{\Lambda}_{(1)}^{-1}$ . The resulting estimator  $\hat{\theta}$  is consistent and asymptotically normal, and is asymptotically efficient within the GMM class (Hansen, 1982). The asymptotic variance also simplifies to  $(\hat{M}^\top \hat{\Lambda}_{(1)}^{-1} \hat{M})^{-1}$ . To further improve finite sample performance, researchers can consider alternative GMM estimators, such as continuously updating GMM (Hansen *et al.*, 1996).

### 3 Double Negative Controls for Network Data

In this section, we consider a sample of interconnected units in a single network. Extending results in Section 2, we propose the double negative control approach to identification of the

ACPE in the presence of unmeasured network confounding, which includes latent homophily and contextual confounding as special cases. We then examine estimation and inference while accounting for both unmeasured network confounding and network-dependent observations.

### 3.1 Notation and Definitions

Suppose one has observed data on a population of  $n$  units interconnected by a network. We let  $N_n = \{1, \dots, n\}$  be the set of unit indices. We consider an undirected network  $\mathcal{G}_n$  where ties or links between units are mutual, and connected units can affect each other. Formally, we define  $\mathcal{G}_n = (N_n, \mathbf{G})$ , i.e., a set of units  $N_n$  connected by mutual ties, represented by a network adjacency matrix  $\mathbf{G}$ . The network adjacency matrix  $\mathbf{G}$  depends on sample size  $n$ , but the index will be suppressed in the following discussion as it eases the exposition without confusion. The entry  $G_{ij}$  takes the value of one if unit  $i$  and  $j$  are connected and takes the value of zero otherwise. We follow the convention that  $G_{ii} = 0$  for  $i \in N_n$ , and we call units  $i$  and  $j$  *peers* if  $G_{ij} = 1$ . We also define two network notations useful throughout the paper. We define network distance  $d_n(i, j)$  to be the length of the shortest path between nodes  $i$  and  $j$  on network  $\mathcal{G}_n$ . We define  $\mathcal{N}_n(i; s)$  to be a set of nodes that are at distance  $s$  from node  $i$ :

$$\mathcal{N}_n(i; s) = \{j \in N_n : d_n(i, j) = s\}.$$

For each  $i \in N_n$ , one observes  $(Y_{i1}, Y_{i2}, \mathbf{X}_i)$ , where  $Y_{it}$  denotes the focal behavior of unit  $i$  at time  $t \in \{1, 2\}$ , and  $\mathbf{X}_i$  are covariates of unit  $i$  measured prior to  $Y_{i1}$ .  $\mathbf{X}_i$  can include network-characteristics, such as the network degree of unit  $i$ . We call  $t = 1$  baseline and  $t = 2$  follow-up.

For the sake of clarity in the exposition, we restrict presentation of all main results to the causal effect from peers. It is important to emphasize that results in this section, however, do not assume the absence of the causal effects from higher-order peers (e.g., peers-of-peers); we only consider such higher-order peer effects as nuisance. In Section 6, we discuss similar results for a general case where higher-order peer effects (e.g., the causal effect from peers-of-peers) is the main causal estimand of interest.

As the outcome variable, we focus on focal behavior at follow-up  $Y_{i2}$ . In principle, it is possible to perform causal inference by defining a multivariate treatment variable based on focal behaviors of peers at baseline  $\{Y_{j1} : j \in \mathcal{N}(i; 1)\}$ . However, in practice, researchers may need to make a dimension-reducing assumption, known as an exposure mapping (Aronow and Samii, 2017), to define the treatment variable,  $A_i = \phi(\{Y_{j1} : j \in \mathcal{N}(i; 1)\}) \in \mathbb{R}$  where function  $\phi$  is specified by a researcher based on subject matter knowledge. For example, the most common choice is  $A_i = \sum_{j=1}^n G_{ij} Y_{j1} / \sum_{j=1}^n G_{ij}$ , while our results can accommodate any choice of  $\phi$ .

The potential outcome  $Y_{i2}(a)$  is defined as the outcome that would realize when the treatment variable is set to  $A_i = a$ . We make the standard consistency assumption linking observed and potential outcomes,  $Y_{i2} = Y_{i2}(A_i)$ , throughout the paper. Our goal is to estimate the Average Causal Peer Effect (ACPE), defined as

$$\tau(a, a') := \frac{1}{n} \sum_{i=1}^n \mathbb{E} \{Y_{i2}(a) - Y_{i2}(a')\} \quad (7)$$

where  $a, a' \in \mathcal{A}$  where  $\mathcal{A}$  is the support of  $A$ . We define the expectation conditional on the observed network  $\mathcal{G}_n$ , while we omit its conditioning for notational simplicity. Unlike typical causal parameters in i.i.d settings, units' potential outcome may not share a common expectation, i.e.,  $\mathbb{E} \{Y_{i2}(a) - Y_{i2}(a')\} \neq \mathbb{E} \{Y_{j2}(a) - Y_{j2}(a')\}$  for  $i \neq j$ . Thus, the causal estimand is explicitly written as the empirical mean.

To identify the ACPE, existing works rely upon the assumption that the observed variables are sufficient to account for confounding of the relationship between  $A_i$  and  $Y_{i2}(a)$ , i.e.,

$$Y_{i2}(a) \perp\!\!\!\perp A_i \mid Y_{i1}, \mathbf{X}_i,$$

where observed pre-treatment covariates  $\mathbf{X}_i$  can include observed covariates of peers of unit  $i$  or covariates of other units who are indirectly connected to unit  $i$ . Even though some recent methods allow for network dependence across units (e.g., Ogburn *et al.*, 2017; Tchetgen Tchetgen *et al.*, 2020b), they assume such latent network dependence does not confound the main outcome-treatment relationship. However, such assumption is, in general, untenable in many applications due to unmeasured network confounding, including unmeasured homophily (Shalizi and Thomas, 2011) and contextual confounding (VanderWeele and An, 2013), as discussed in Section 2.2.

### 3.2 Identification Assumptions with Double Negative Controls

We propose an alternative approach based on double negative controls. We generalize Assumptions 1.1–1.4 in Section 2 to the network setting.

#### Assumption 2

1. (Latent Ignorability). *For all  $a \in \mathcal{A}$  and all  $i \in N_n$ ,*

$$Y_{i2}(a) \perp\!\!\!\perp A_i \mid U_i, \mathbf{X}_i.$$

2. (Negative Controls). *For all  $i \in N_n$ ,*

$$\text{(Negative Control Outcome)} \quad W_i \perp\!\!\!\perp A_i \mid U_i, \mathbf{X}_i,$$



(Negative Control Exposure)  $\mathbf{Z}_i \perp\!\!\!\perp Y_{i2} \mid A_i, U_i, \mathbf{X}_i$ , and  $\mathbf{Z}_i \perp\!\!\!\perp W_i \mid A_i, U_i, \mathbf{X}_i$ ,

where  $\mathbf{Z}_i$  is a vector of negative control exposures.

3. (Outcome Confounding Bridge). *There exists an outcome confounding bridge function  $h(W_i, A_i, \mathbf{X}_i)$  such that for all  $a \in \mathcal{A}$ , and all  $i \in N_n$ ,*

$$\mathbb{E}(Y_{i2} \mid A_i = a, U_i, \mathbf{X}_i) = \mathbb{E}\{h(W_i, a, \mathbf{X}_i) \mid A_i = a, U_i, \mathbf{X}_i\}. \quad (8)$$

4. (Negative Control Relevance). *For any square integrable function  $f$  and any  $a$  and  $\mathbf{x}$ , if  $\mathbb{E}\{f(W_i) \mid \mathbf{Z}_i = \mathbf{z}, A_i = a, \mathbf{X}_i = \mathbf{x}\} = 0$  for almost all  $\mathbf{z}$ , then  $f(W_i) = 0$  almost surely.*

### 3.3 Selecting Negative Controls in Network Settings

In the proposed double negative control approach, selection of negative control outcome and exposure is essential in practice. While, in principle any negative controls satisfying Assumption 2 can be used, we discuss two convenient strategies to select negative controls in the network setting.

#### 3.3.1 Using Auxiliary Variables

First, as in Section 2, a plausible candidate is an auxiliary variable  $C_i$  that (a) does not affect network relationships  $\mathcal{G}_n$  and (b) does not affect variables of other units. For example, in the context of Add Health data, whether a student experiences headaches is likely to satisfy this condition (See “Example (Negative Controls)” in Section 2.3).

Figure 3 represents examples of causal graphs where Assumption 2.2 holds. We view unit 2 as an ego, who has two peers (units 1 and 3) and one peer-of-peers (unit 4). We use a fully connected chain graph (Lauritzen and Richardson, 2002) to denote general network dependence of latent confounders  $U$  across units. This chain graph representation is one general way to capture unmeasured network confounding, which can accommodate both unmeasured homophily and contextual confounding.

In Figure 3.(i),  $C_2$  satisfies NCO conditions and three variables  $\{C_1, C_3, C_4\}$  satisfy NCE conditions. Importantly, not only auxiliary variables of peers but also those of peers-of-peers also satisfy the conditions of the NCE.

Finally, we provide primitive sufficient conditions that imply the negative control conditions. In particular, the following conditions capture a general approach for using auxiliary variables as negative controls.

$$C_i \perp\!\!\!\perp \{(C_j, Y_{j1}) : j \neq i\} \mid U_i, \mathbf{X}_i, \quad (9)$$

$$Y_{i2} \perp\!\!\!\perp \{C_j : j \neq i\} \mid A_i, U_i, \mathbf{X}_i, \quad (10)$$

Equations (9) and (10) formalize the notion that  $C_i$  should not affect network relationships and should not affect peers' variables. Lemma 2 below shows that auxiliary variable  $C_i$  can serve as a valid negative control if it satisfies the stated conditions (Figure 3.(i) is an example).

**Lemma 2** *Suppose auxiliary variable  $C_i$  satisfies the two conditions (equations (9) and (10)). Then, Assumption 2.2 holds with  $W_i = C_i$  and  $\mathbf{Z}_i = \{C_j : j \neq i\}$ .*

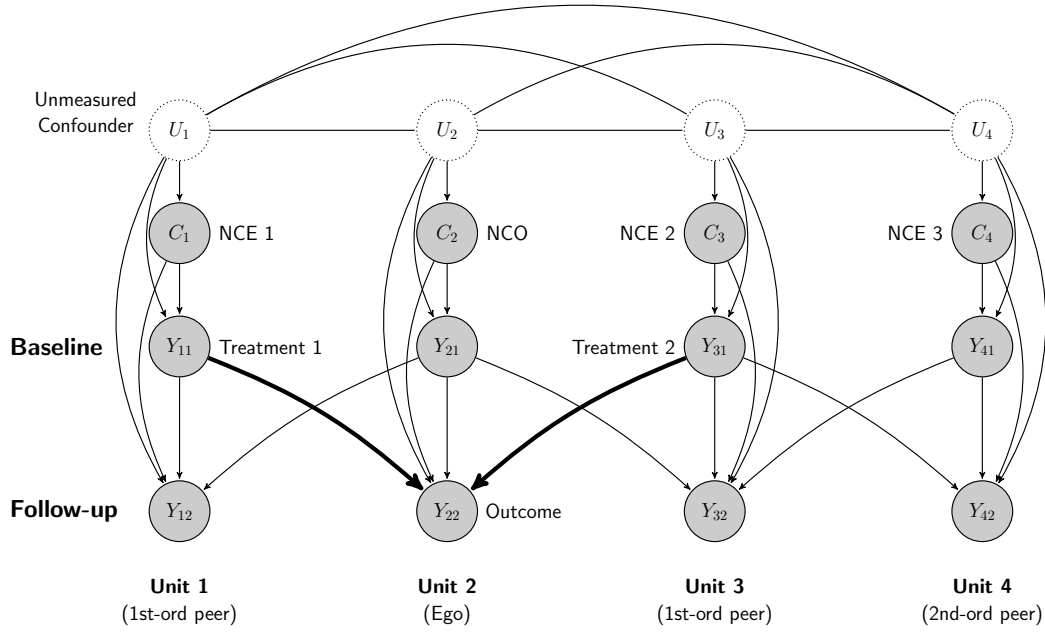
Several points are worth noting. First, these are sufficient conditions, not necessary conditions for Assumption 2.2. Therefore, any negative controls that satisfy Assumption 2 can in principle be used for identification. Second, Lemma 2 suggests that in practice, there may be multiple NCEs because one can use auxiliary variables of all other units  $\{C_j : j \neq i\}$ . Therefore, it may be possible to enhance identification and increase estimation efficiency by exploiting a large number of NCEs. We plan to examine optimal selection and specification of NCEs in future work.

### 3.3.2 Using the Focal Behaviors

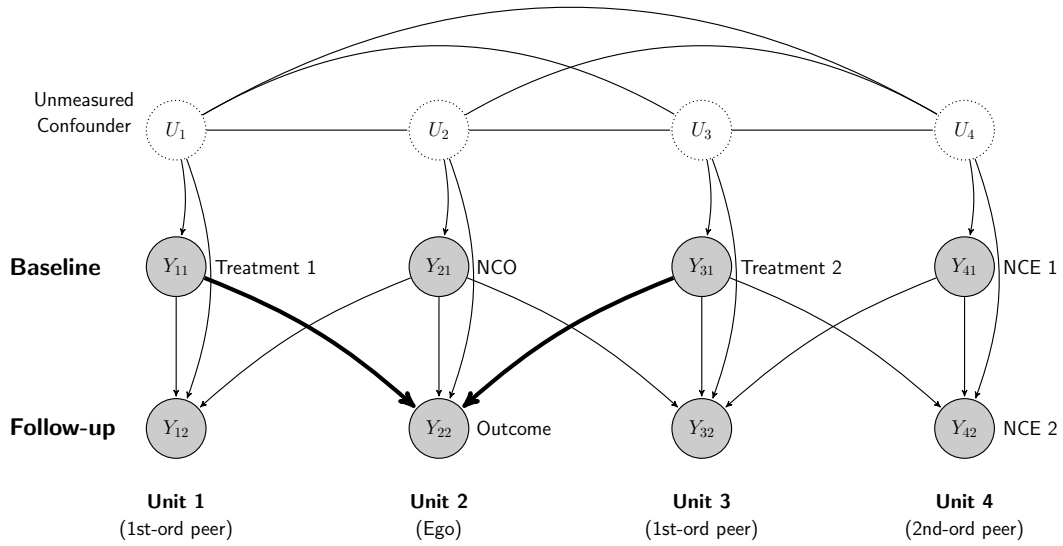
Second, in certain settings, we may also use focal behaviors  $Y_{it}$  of peers and those measured at baseline as plausible candidates for negative controls. In particular, ego's focal behavior at baseline may serve as valid NCO and focal behaviors of peers-of-peers  $\{Y_{jt} : j \in \mathcal{N}(i; 2), t \in \{1, 2\}\}$  may constitute valid NCEs. Figure 3.(ii) represents a causal graph illustrating an instance of the causal model where  $Y_{21}$  qualifies as NCO and variables  $\{Y_{41}, Y_{42}\}$  qualify as NCE. More generally, when focal behaviors of peers-of-peers constitute valid negative control exposures, focal behaviors of units at least of network distance 2 from node  $i$  may be credible negative control exposures. A hybrid approach might entail combining the auxiliary variables and focal behaviors as negative controls. In Figure 3.(i), we define focal behavior measured at baseline  $Y_{21}$  as NCO (instead of  $C_2$ ) and auxiliary variables of peers  $\{C_1, C_3, C_4\}$  as NCEs.

This selection of negative controls is particularly plausible when focal behaviors do not have direct causal relationships with peers' focal behaviors measured concurrently. In Figure 3.(ii), while peers' focal behaviors measured at baseline affect egos' focal behaviors measured at follow-up (e.g.,  $Y_{11}$  and  $Y_{31}$  affect  $Y_{22}$ ), peers' focal behaviors cannot causally affect egos' focal behaviors measured concurrently (e.g.,  $Y_{11}$  and  $Y_{31}$  do not affect  $Y_{21}$ ;  $Y_{12}$  and  $Y_{32}$  do not affect  $Y_{22}$ ). This absence of causal simultaneity has previously been assumed in the literature of causal peer effects (Shalizi and Thomas, 2011; Ogburn and VanderWeele, 2014; Egami, 2018; Liu and Tchetgen Tchetgen, 2020; McFowland III and Shalizi, 2021).

In practice, this assumption is most credible when researchers a priori know that the focal behaviors of units are indeed measured concurrently. For example, in Add Health data, students'



(i)



(ii)

Figure 3: Examples of chain graphs with Double Negative Controls. *Note:* We use fully connected chain graphs to denote general network dependence of latent confounders  $U$  across units. For concreteness, we show Unit 2 as the ego. The thick arrows from  $Y_{11}$  to  $Y_{22}$  and from  $Y_{31}$  to  $Y_{22}$  indicate the causal peer effects of interest. We use shaded (dotted) nodes to denote observed (unobserved) variables.

GPA are likely to be measured at the same time for students within a school, and thus, a student’s GPA cannot be affected by peers’ GPA in the same semester. Importantly, a student’s GPA can be affected by peers’ GPA in the last semester, and a student’s study habit might be affected by peer’s study habits within the same semester. These, however, do not invalidate the use of GPA of peers-of-peers as NCEs as long as students’ GPA within the same semester do not have direct causal relationships with each other. In some applications, analysts can directly measure focal behaviors of interest. In such cases, by virtue of survey/study design, researchers can ensure that the focal behaviors measured at each wave do not affect peers’ focal behaviors within the same wave by conducting surveys concurrently. This assumption is less credible when measurements of focal behaviors are aggregated over long periods of time, such as the number of political tweets over a year, which is likely to be affected by peers’ tweets within the same year. This is often called the temporal aggregation problem, which invalidates not only peer effect analysis but also a large class of panel data analyses (Granger, 1988).

As in Section 3.3.1, we provide primitive sufficient conditions for valid negative controls. In particular, the following conditions capture a general approach for leveraging focal behaviors as negative controls.

$$Y_{i1} \perp\!\!\!\perp \{Y_{j1} : j \neq i\} \mid U_i, \mathbf{X}_i, \quad (11)$$

$$Y_{i2} \perp\!\!\!\perp \{Y_{j1} : j \in \mathcal{N}(i; s), s \geq \tilde{s}\} \mid A_i, U_i, \mathbf{X}_i, \quad \text{with some integer } \tilde{s}, \quad (12)$$

Equation (11) formalizes the notion that focal behaviors do not have direct causal relationships with the peers’ focal behaviors measured at the same time. Equation (12) captures the assumption that only peers closer than distance  $s$  can have causal peer effects on an ego. While we focus on the causal effect from peers as the causal estimand, we do not necessarily need to assume the absence of higher-order peer effects.

Lemma 3 below establishes that focal behaviors can serve as valid negative controls when they satisfy the stated conditions (Figure 3.(ii) is an example).

**Lemma 3** *Suppose that focal behaviors satisfy conditions (11) and (12). Then, Assumption 2.2 holds with  $W_i = Y_{i1}$  and  $\mathbf{Z}_i = \{Y_{j1} : j \in \mathcal{N}(i; s), s \geq \tilde{s}\}$ .*

Again, we emphasize that these conditions are sufficient for Assumption 2.2, and thus, there may be other ways to justify negative control conditions. Any negative controls that satisfy Assumption 2 may be used for identification of ACPE.

This particular selection strategy of negative controls has two advantages when valid. First, when the NCO entails focal behaviors measured at baseline, the confounding bridge assumption (Assumption 2.3) is often more likely to hold because the NCO and the main outcome

are measured on the same scale (Sofer *et al.*, 2016). Second, if researchers can leverage focal behaviors of peers and those measured at baseline as negative controls, researchers do not need to collect additional auxiliary variables, which lowers data collection requirements and improves applicability of the double negative control approach. However, selection of valid negative control variables must always be based on reliable domain knowledge because Assumption 2.1 – Assumption 2.4 must be met.

### 3.4 Nonparametric Identification

Analogous to Theorem 1, we now establish nonparametric identification of the ACPE under Assumption 2.

**Theorem 2** *Under Assumption 2, the confounding bridge function is identified as the unique solution to the following equation.*

$$\mathbb{E}(Y_{i2} \mid \mathbf{Z}_i, A_i, \mathbf{X}_i) = \mathbb{E}\{h(W_i, A_i, \mathbf{X}_i) \mid \mathbf{Z}_i, A_i, \mathbf{X}_i\},$$

and ACPE is identified by

$$\tau(a, a') = \frac{1}{n} \sum_{i=1}^n \mathbb{E}\{h(W_i, a, \mathbf{X}_i) - h(W_i, a', \mathbf{X}_i)\}.$$

We provide a proof in Section A.5 of the supplementary material.

Despite the complexity of network data, the identification assumptions and formula of the ACPE are remarkably similar to those in the dyadic study design. Clearly, selection possibilities of negative controls are more flexible in the network setting. As discussed in Section 3.3, focal behaviors of peers may be used as NCEs in some network applications in addition to auxiliary variables. Thus, from a perspective of causal identification, network data present richer opportunities for NC adjustment than dyadic data in that they offer more options of credible negative controls. An important difference from the dyadic case emerges in estimation and inference where one must appropriately account for network-dependence, a challenging task we consider next.

### 3.5 Estimation and Inference

In this section, we consider estimation and inference for the ACPE while allowing for network-dependent observations.

We define a triangular array of  $\mathbb{R}^v$ -valued random vector,  $\mathbf{L}_{n,i} = (Y_{i2}, A_i, W_i, \mathbf{Z}_i, \mathbf{X}_i)$  for  $i \in N_n$ , adapted to a network  $\mathcal{G}_n$  where  $v$  is the length of the vector  $\mathbf{L}_{n,i}$ . Similar to Section 2,

we define a moment estimating function with parameter  $\theta = (\tau, \gamma)$ .

$$m(\mathbf{L}_{n,i}; \theta) = \begin{Bmatrix} \tau - \{h(W_i, a, \mathbf{X}_i; \gamma) - h(W_i, a', \mathbf{X}_i; \gamma)\} \\ \{Y_{i2} - h(W_i, A_i, \mathbf{X}_i; \gamma)\} \times \eta(A_i, \mathbf{Z}_i, \mathbf{X}_i) \end{Bmatrix}.$$

The GMM estimator for  $\theta$  is

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \bar{m}(\theta)^\top \Omega \bar{m}(\theta), \quad (13)$$

where  $\bar{m}(\theta) = \frac{1}{n} \sum_{i=1}^n m(\mathbf{L}_{n,i}; \theta)$  and  $\Omega$  is a user-specified positive-definite weight matrix. Therefore, the proposed DNC estimator  $\hat{\tau}(a, a')$  for  $\tau(a, a')$  is the first element of  $\hat{\theta}$  defined in equation (13). Asymptotic results we derive below hold for the two-step GMM or other alternative GMM estimators (Hansen *et al.*, 1996), too.

Since we consider a sample of interconnected units in a network, the assumption that  $m(\mathbf{L}_{n,i}; \theta)$  is independently and identically distributed is unrealistic. Below, we consider assumptions on the observed data law that are considerably weaker, but still allow for valid inferences about the ACPE in network settings. For ease of exposition, we consider a setting in which the expected causal peer effect,  $\mathbb{E}\{Y_{i2}(a) - Y_{i2}(a')\}$ , is constant across units, while otherwise allowing for network-dependent (i.e., non-independent) errors. In the supplementary material, we extend our results to more general settings of heterogeneous ACPE.

We define  $\psi$ -network dependence (Kojevnikov *et al.*, 2020), which encodes the degree of stochastic dependence between variables in terms of network distance.

**Definition 1 ( $\psi$ -Network Dependence (Kojevnikov *et al.*, 2020))** *The triangular array  $\{\mathbf{V}_{n,i}\}_{i \in N_n}, n \geq 1, \mathbf{V}_{n,i} \in \mathbf{R}^v$ , is called conditionally  $\psi$ -weakly dependent given  $\mathcal{G}_n$ , if for each  $n \in \mathbb{N}$ , there exist a  $\mathcal{G}_n$ -measurable sequence  $\beta_n = \{\beta_{n,s}\}, s \geq 0, \beta_{n,0} = 1$ , and a collection of nonrandom functions  $(\psi_{q_1, q_2})_{q_1, q_2 \in \mathbb{N}}, \psi_{q_1, q_2} : \mathcal{L}_{v, q_1} \times \mathcal{L}_{v, q_2} \rightarrow [0, \infty)$ , such that for all  $(Q_1, Q_2) \in \mathcal{P}_n(q_1, q_2; s)^1$  with  $s > 0$  and all  $f_1 \in \mathcal{L}_{v, q_1}$  and  $f_2 \in \mathcal{L}_{v, q_2}$ ,<sup>2</sup>*

$$|\operatorname{Cov}(f_1(\mathbf{V}_{n, Q_1}), f_2(\mathbf{V}_{n, Q_2}) \mid \mathcal{G}_n)| \leq \psi_{q_1, q_2}(f_1, f_2) \beta_{n,s} \quad a.s. \quad (14)$$

*In this case, we call the sequence  $\beta_n = \{\beta_{n,s}\}_{s=1}^\infty$  the weak dependent coefficients of  $\{\mathbf{V}_{n,i}\}_{i \in N_n}$ .*

Coefficient  $\beta_{n,s}$  captures network dependence between units that are at network distance greater than or equal to  $s$  in network  $\mathcal{G}_n$  by the covariance of nonlinearly transformed variables. Thus, a sequence of coefficients  $\beta_n = \{\beta_{n,s}\}_{s=1}^\infty$  captures how fast network dependence between units decays as network distance  $s$  increases. Assumption 3, which we will introduce next, restricts

<sup>1</sup> $\mathcal{P}_n(q_1, q_2; s)$  denotes the collection of two sets of nodes of size  $q_1$  and  $q_2$  with distance between each other of at least  $s$ . Formally,  $\mathcal{P}(q_1, q_2; s) = \{(Q_1, Q_2) : Q_1, Q_2 \subset N_n, |Q_1| = q_1, |Q_2| = q_2, \text{ and } d_n(Q_1, Q_2) \geq s\}$ .

<sup>2</sup> $\mathcal{L}_{v, q_1}$  and  $\mathcal{L}_{v, q_2}$  denote the collection of bounded Lipschitz real functions on  $\mathbf{R}^{v \times q_1}$  and  $\mathbf{R}^{v \times q_2}$ , respectively.

the rate by which this network dependence decays. Importantly, the  $\psi$ -network dependence permits dependence between units  $i$  and  $j$  that are only indirectly connected in the network, and thus, any two units can be dependent as long as there is a network path between them. This is in contrast to two other popular approaches; (1) dependency graphs, which can allow units to be dependent only when they are adjacent in a given network, and (2) Markov random fields, which impose conditional independence restrictions based on the network structure (e.g., a given unit's observed data are independent of data observed for all units in the network conditional on observed data for its first-order network peers).

Using the notion of  $\psi$ -network dependence, we make the following assumptions on the observed data distribution that permit network dependent error, but still allow for making inferences about the ACPE.

**Assumption 3** *The triangular arrays  $\{m(\mathbf{L}_{n,i}; \theta)\}_{i \in N_n, n \geq 1}$  and  $\{\frac{\partial}{\partial \theta} m(\mathbf{L}_{n,i}; \theta)\}_{i \in N_n, n \geq 1}$  are conditionally  $\psi$ -weakly dependent given  $\mathcal{G}_n$ , respectively, for all  $\theta \in \Theta$  with the weak dependent coefficients  $\beta_n$  that satisfy the following conditions.  $\sup_{n \geq 1} \max_{s \geq 1} \beta_{n,s} < \infty$ , and for some constant  $\lambda$ ,  $\psi_{q_1, q_2}(f_1, f_2) \leq \lambda \times q_1 q_2 (\|f_1\|_\infty + \text{Lip}(f_1)) (\|f_2\|_\infty + \text{Lip}(f_2))$ .<sup>3</sup> There exist  $p > 4$  and a sequence  $\xi_n \rightarrow \infty$  such that*

1.  $\beta_{n, \xi_n}^{(p-1)/p} = o_{a.s.}(n^{-3/2})$ ,
2. for each  $k \in \{1, 2\}$ ,

$$\frac{1}{n^{k/2}} \sum_{s \geq 0} r_n(s, \xi_n; k) \beta_{n,s}^{1-(k+2)/p} = o_{a.s.}(1)$$

where we define the neighborhood shell:

$$r_n(s, \xi_n; k) = \inf_{\alpha > 1} \left\{ \frac{1}{n} \sum_{i \in N_n} \max_{j \in \tilde{\mathcal{N}}_n(i; s)} |\tilde{\mathcal{N}}_n(i; \xi_n) \setminus \tilde{\mathcal{N}}_n(j; s-1)|^{k\alpha} \right\}^{\frac{1}{\alpha}} \times \left\{ \frac{1}{n} \sum_{i \in N_n} |\mathcal{N}_n(i; s)|^{\frac{\alpha}{\alpha-1}} \right\}^{1-\frac{1}{\alpha}}, \quad (15)$$

and the within- $s$  peers  $\tilde{\mathcal{N}}_n(i; s) = \{j \in N_n : d_n(i, j) \leq s\}$ . We use  $|\cdot|$  to denote the cardinality of a set.

This is an adaptation of Condition ND in Kojevnikov *et al.* (2020) to our setup. Assumption 3.1 restricts the speed by which weak dependent coefficients  $\beta_{n,s}$  decay as network distance  $s$  increases. Assumption 3.2 restricts the speed by which the density of the network changes as sample size increases. When network dependence  $\beta_{n,s}$  decays faster with network distance  $s$ , it

<sup>3</sup>Lip( $f$ ) represents Lipschitz constant of  $f$ , and  $\|\cdot\|_\infty$  denotes the sup norm, i.e.,  $\|f\|_\infty = \sup |f(x)|$ .

can accommodate denser networks. See Kojevnikov *et al.* (2020) for further discussion on these conditions.

We are now ready to state asymptotic properties of the proposed DNC estimator.

**Theorem 3** *Under the conditions given in Theorem 2, Assumption 3 and standard GMM regularity conditions,<sup>4</sup> as  $n$  goes to infinity, the DNC estimator is consistent:*

$$\widehat{\tau}(a, a') \xrightarrow{p} \tau(a, a'),$$

and asymptotically normal:

$$\frac{\widehat{\tau}(a, a') - \tau(a, a')}{\sqrt{\sigma^2/n}} \xrightarrow{d} \text{Normal}(0, 1).$$

The asymptotic variance  $\sigma^2$  is the (1,1) th element of matrix  $\Sigma$  where

$$\begin{aligned} \Sigma &= \Gamma_0 \Lambda_0 \Gamma_0^\top, \quad \Lambda_0 = \text{Var} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n m(\mathbf{L}_{n,i}; \theta_0) \right) \\ \Gamma_0 &= (M_0^\top \Omega M_0)^{-1} M_0^\top \Omega, \quad M_0 = \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left\{ \frac{\partial}{\partial \theta} m(\mathbf{L}_{n,i}; \theta_0) \right\}, \end{aligned}$$

and we define  $\theta_0$  to be the true parameter such that, for all units  $i \in N_n$ ,  $\mathbb{E} \{m(\mathbf{L}_{n,i}; \theta)\} = 0$  only when  $\theta = \theta_0$ .

We provide a proof in Section B of the supplementary material.

To estimate the standard error of the DNC estimator, the key is to estimate  $\Lambda_0$ . To account for network-dependent errors, we rely on the network HAC variance estimator (Kojevnikov *et al.*, 2020) adapted to our setting:

$$\widehat{\Lambda} = \sum_{s \geq 0} \omega(s/b_n) \left\{ \frac{1}{n} \sum_{i \in N_n} \sum_{j \in \mathcal{N}_n(i;s)} m(\mathbf{L}_{n,i}; \widehat{\theta}) m(\mathbf{L}_{n,j}; \widehat{\theta})^\top \right\}, \quad (16)$$

where a kernel function  $\omega(\cdot)$  is defined as follows:  $\omega : \overline{\mathbb{R}} \rightarrow [-1, 1]$  such that  $\omega(0) = 1, \omega(c) = 0$  for  $|c| \geq 1$  and  $\omega(c) = \omega(-c)$  for all  $c \in \overline{\mathbb{R}}$ . Examples include the truncated, Parzen, and Tukey–Hanning kernels.  $b_n$  denotes a bandwidth of the network HAC variance estimator. This bandwidth determines how far  $\widehat{\Lambda}$  takes into account network dependence; kernel weight  $\omega(s/b_n) > 0$  for  $s < b_n$  and  $\omega(s/b_n) = 0$  for  $s \geq b_n$ .

The variance estimator of the DNC estimator can be computed as the (1,1) th element of matrix  $\widehat{\Sigma}$  defined as

$$\widehat{\Sigma} = \widehat{\Gamma} \widehat{\Lambda} \widehat{\Gamma}^\top \quad (17)$$

---

<sup>4</sup>In the supplementary material, we provide the regularity conditions widely used in the GMM framework (Hansen, 1982; Newey and McFadden, 1994).



where  $\widehat{\Gamma} = (\widehat{M}^\top \Omega \widehat{M})^{-1} \widehat{M}^\top \Omega$ , and  $\widehat{M} = \frac{1}{n} \sum_{i=1}^n \frac{\partial}{\partial \theta} m(\mathbf{L}_{n,i}; \widehat{\theta})$ .

We now establish consistency of this network HAC variance estimator. The following result formally restricts the speed by which bandwidth  $b_n$  should increase as sample size increases. When network dependence  $\beta_{n,s}$  decays slower and the average number of network peers at distance  $s$  increases faster with network distance, bandwidth  $b_n$  should increase faster as sample size increases.

**Theorem 4** *Under the conditions given in Theorem 3, suppose the choice of bandwidth and kernel satisfies the following condition with  $p$  that satisfies Assumption 3.*

$$\lim_{n \rightarrow \infty} \sum_{s \geq 0} |\omega(s/b_n) - 1| \rho_n(s) \beta_{n,s}^{1-2/p} = 0 \quad a.s., \quad (18)$$

where  $\rho_n(s)$  is the average number of network peers at the distance  $s$ ,  $\rho_n(s) = \frac{1}{n} \sum_{i=1}^n |\mathcal{N}_n(i; s)|$ . Then,  $\widehat{\Sigma} \xrightarrow{p} \Sigma$ .

We provide a proof in Section B of the supplementary material.

In practice, we recommend using the default choice of bandwidth  $b_n$  provided by Kojevnikov *et al.* (2020), i.e.,

$$b_n = \text{constant} \times \frac{\log(n)}{\log\{\max(\text{average degree}, 1.05)\}}. \quad (19)$$

In our simulation studies (Section 4), we set the constant in equation (19) to 1.0 and find this default choice performs well across various settings.

Finally, under conditions given in Theorem 4, we obtain an asymptotically valid  $(1 - \alpha)$  confidence interval for  $\tau(a, a')$  by  $[\widehat{\tau}(a, a') - \Phi(1 - \alpha/2) \times \widehat{\sigma}/\sqrt{n}, \widehat{\tau}(a, a') + \Phi(1 - \alpha/2) \times \widehat{\sigma}/\sqrt{n}]$  where  $\widehat{\sigma}$  is the  $(1, 1)$ th element of matrix  $\widehat{\Sigma}$  defined in equation (17).

### 3.6 Linear Double Negative Control Estimator

Here, we discuss an important special case under a linear specification for the confounding bridge function, which admits a closed form solution. Suppose we assume a linear confounding bridge:

$$h(W_i, A_i, \mathbf{X}_i; \gamma) = \gamma_\alpha + \gamma_A A_i + \gamma_W W_i + \gamma_X^\top \mathbf{X}_i.$$

Under this linear model,  $\tau(a, a') = \gamma_A \times (a - a')$ . We can estimate coefficients  $\gamma = (\gamma_\alpha, \gamma_A, \gamma_W, \gamma_X^\top)^\top$  by fitting the linear GMM estimator:

$$\widehat{\gamma} = (\mathbf{V}_W^\top \mathbf{V}_Z \Omega \mathbf{V}_Z^\top \mathbf{V}_W)^{-1} \mathbf{V}_W^\top \mathbf{V}_Z \Omega \mathbf{V}_Z^\top \mathbf{Y} \quad (20)$$

where  $\mathbf{V}_W$  is a matrix with  $n$  rows with  $i$ th row  $\mathbf{V}_{iW} = (1, A_i, W_i, \mathbf{X}_i^\top)^\top$ ,  $\mathbf{V}_Z$  is a matrix with  $n$  rows with  $i$ th row  $\mathbf{V}_{iZ} = (1, A_i, \mathbf{Z}_i^\top, \mathbf{X}_i^\top)^\top$ , and  $\mathbf{Y}$  is a  $n$ -dimensional vector with  $i$ th element equal to  $Y_{i2}$ .

To account for network dependence of samples, we adopt the network HAC variance estimator in Section 3.5. The key is to estimate

$$\Lambda_0^{\text{lin}} = \text{Var} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n e_{i2} \mathbf{V}_{iZ} \right), \quad \text{and} \quad e_{i2} = Y_{i2} - \gamma^\top \mathbf{V}_{iW}. \quad (21)$$

Using the network HAC variance estimator, we can estimate the variance of  $\hat{\gamma}$  as

$$\widehat{\text{Var}}(\hat{\gamma}) = n(\mathbf{V}_W^\top \mathbf{V}_Z \Omega \mathbf{V}_Z^\top \mathbf{V}_W)^{-1} \mathbf{V}_W^\top \mathbf{V}_Z \Omega \widehat{\Lambda}^{\text{lin}} \Omega \mathbf{V}_Z^\top \mathbf{V}_W (\mathbf{V}_W^\top \mathbf{V}_Z \Omega \mathbf{V}_Z^\top \mathbf{V}_W)^{-1},$$

where

$$\widehat{\Lambda}^{\text{lin}} = \sum_{s \geq 0} \omega(s/b_n) \left\{ \frac{1}{n} \sum_{i \in N_n} \sum_{j \in \mathcal{N}_n(i; s)} \widehat{e}_{i2} \widehat{e}_{j2} \mathbf{V}_{iZ} \mathbf{V}_{jZ}^\top \right\}, \quad \text{and} \quad \widehat{e}_{i2} = Y_{i2} - \widehat{\gamma}^\top \mathbf{V}_{iW}.$$

Researchers can use the two-step GMM to minimize the asymptotic variance within this class. In the first step, we set  $\Omega = (\mathbf{V}_Z^\top \mathbf{V}_Z)^{-1}$ , and compute a preliminary GMM estimate  $\widehat{\gamma}_{(1)}$ . Importantly, one may estimate  $\widehat{\gamma}_{(1)}$  using any off-the-shelf software package for two-stage least squares, such as `ivreg` in R, by viewing  $W_i$  as the endogenous treatment,  $\mathbf{Z}_i$  as the instrument, and  $(A_i, \mathbf{X}_i^\top)$  as covariates (see Tchetgen Tchetgen *et al.*, 2020a). Here we use the two-stage least squares as a convenient way to compute this preliminary GMM estimate, and thus, we do not make any assumptions required for standard instrumental variable analysis. See Miao *et al.* (2018a) for relationships between the double negative control approach and the instrumental variable approach in general. In the second step, we compute  $\widehat{\Lambda}^{\text{lin}}$  based on  $\widehat{\gamma}_{(1)}$ , which is denoted by  $\widehat{\Lambda}_{(1)}^{\text{lin}}$ . Then, we obtain the final estimator by solving equation (20) with  $\Omega = (\widehat{\Lambda}_{(1)}^{\text{lin}})^{-1}$ . The resulting estimator  $\hat{\gamma}$  is consistent and asymptotically normal, and is asymptotically efficient within the GMM class under the conditions given in Theorems 3 and 4. The variance also simplifies to

$$\widehat{\text{Var}}(\hat{\gamma}) = n(\mathbf{V}_W^\top \mathbf{V}_Z (\widehat{\Lambda}^{\text{lin}})^{-1} \mathbf{V}_Z^\top \mathbf{V}_W)^{-1}.$$

### Choice of Bandwidth

In general settings of network-dependent errors (Section 3.5), one must estimate a bandwidth  $b_n$  for the network HAC variance estimator (e.g., Kojevnikov *et al.*, 2020) because how far network dependence persists is a priori unknown. However, when the following assumption holds, we can analytically select the bandwidth.

### Assumption 4

1. The ACPE is equal to a linear function of parameters  $\gamma$  in the confounding bridge function.

2. There exists integer  $s^*$  such that for units  $i, j$  with distance  $d_n(i, j) \geq s^*$ ,

$$\mathbf{L}_{n,j} \perp\!\!\!\perp \mathbf{L}_{n,i} \mid A_i, \mathbf{Z}_i, \mathbf{X}_i, U_i.$$

Assumption 4.1 holds for a linear confounding bridge function as we consider in this section. Assumption 4.2 requires that observed data for unit  $j$ ,  $\mathbf{L}_{n,j}$ , is conditionally independent of observed data for unit  $i$ ,  $\mathbf{L}_{n,i}$ , given unit  $i$ 's treatment, NCEs, observed pre-treatment covariates, and the unmeasured confounder. This conditional independence is required only upon conditioning on latent confounder  $U_i$ , and thus, it does not restrict network dependence of the observed data law itself.

Importantly, we emphasize that Assumption 4.2 holds under many relevant scenarios. Figure 3 provides examples of causal graphs where Assumption 4.2 is satisfied. In Figure 3.(i), suppose one uses  $C_i$  as the NCO and  $\mathbf{Z}_i = \{C_j : j \in \mathcal{N}_n(i; 1)\}$  as the NCEs. Then, Assumption 4.2 holds with  $s^* = 2$ . If one uses auxiliary variables of both peers and peers-of-peers,  $\mathbf{Z}_i = \{C_j : j \in \{\mathcal{N}_n(i; 1), \mathcal{N}_n(i; 2)\}\}$ , Assumption 4.2 holds with  $s^* = 3$ . Figure 3.(ii) represents another example. Suppose one exploits  $Y_{i1}$  as the NCO and  $\mathbf{Z}_i = \{Y_{jt} : j \in \mathcal{N}_n(i; 2), t \in \{1, 2\}\}$  as the NCEs. Then, Assumption 4.2 holds with  $s^* = 4$ .

Under Assumption 4, Lemma 4 below shows that one can analytically select the bandwidth for the network HAC variance estimator.

**Lemma 4** *Suppose the conditions given in Theorem 3 hold. Under Assumption 4.1, we can simplify the moment function to  $\tilde{m}(\mathbf{L}_{n,i}; \gamma) = \{Y_{i2} - h(W_i, A_i, \mathbf{X}_i; \gamma)\} \times \eta(A_i, \mathbf{Z}_i, \mathbf{X}_i)$  as our target parameter is a linear function of  $\gamma$ . Then, under Assumption 4.2 with integer  $s^*$ , we can use the following network HAC variance estimator for  $\hat{\gamma}$ , which is the GMM estimator with moment function  $\tilde{m}(\mathbf{L}_{n,i}; \gamma)$ .*

$$\widehat{\text{Var}}(\hat{\gamma}) = \frac{1}{n} \widehat{\Gamma}_\gamma \widehat{\Lambda}_{s^*} \widehat{\Gamma}_\gamma^\top \quad (22)$$

where

$$\widehat{\Lambda}_{s^*} = \sum_{s=0}^{s^*-1} \omega(s/b_n) \left\{ \frac{1}{n} \sum_{i \in N_n} \sum_{j \in \mathcal{N}_n(i; s)} \tilde{m}(\mathbf{L}_{n,i}; \hat{\gamma}) \tilde{m}(\mathbf{L}_{n,j}; \hat{\gamma})^\top \right\},$$

$$\widehat{\Gamma}_\gamma = (\widehat{M}_\gamma^\top \Omega \widehat{M}_\gamma)^{-1} \widehat{M}_\gamma^\top \Omega, \text{ and } \widehat{M}_\gamma = \frac{1}{n} \sum_{i=1}^n \frac{\partial}{\partial \gamma} \tilde{m}(\mathbf{L}_{n,i}; \hat{\gamma}).$$

The key to this result is that, to compute the variance of a sum of products of moments, one only needs to consider moments of units with distance less than  $s^*$  (i.e., we added  $s \in \{0, 1, \dots, s^* - 1\}$ ), as the remaining contributions are null. This is in contrast to the default network HAC variance estimator (equation (16)) where we have to incorporate all products of

moments of units with distance less than  $b_n$ , which is in general larger than  $s^*$ . We provide a proof in Section B of the supplementary material.

For the linear DNC estimator, Assumption 4.1 automatically holds, and thus, as long as Assumption 4.2 holds, we can rely on this analytical choice of bandwidth. We evaluate both analytical and default bandwidth selections (equation (19)) in a simulation study (Section 4).

## 4 Simulation Study

We investigate the finite sample performance of the proposed DNC estimator of the ACPE using networks of varying density and size. In the supplementary material, we also examine the performance of the proposed estimator in settings where key identification assumptions are violated; in Section C.1, we consider violation of the negative control assumption (Assumption 2.2), and in Section C.2, we examine violation of the outcome confounding bridge assumption (Assumption 2.3) due to violation of the underlying completeness condition.

**Setup.** To investigate the performance of the proposed estimator, we consider two different types of networks: the small world network and the real-world network from Add Health data. To generate the small world network, we use `sample_smallworld` with the rewiring probability of 0.15 based on R package `igraph`. We consider two levels of densities: low (the average degree of four) and high (the average degree of eight). Add Health project collected detailed information about friendship networks by an in-school survey. We define friendships as symmetric relationships: the pair of students  $i$  and  $j$  in the same school are coded as friends if either  $i$  lists  $j$  as a friend, or  $j$  lists  $i$  as a friend, or both. While we analyze this data more thoroughly in Section 5, we also use it here as basis for the simulation. For each simulation, we generate a network of size  $n$  where we consider sample size  $n \in \{500, 1000, 2000, 4000\}$ . For the small-world network, we generate a single network of size  $n$ . For the Add Health network, we retain the original network characteristics by randomly sampling schools with probability proportional to its size until the total sample size reaches  $n$ . The average degree of the Add Health network ranges from 3.82 to 5.95, and its average number of the peers-of-peers ranges from 20.85 to 33.24, both of which are in the middle of the low-density small-world network (average degree = 4) and the high-density small world network (average degree = 8). The density of the Add Health network ranges from 0.15 to 0.77 %, which are close to the density of the low-density small-world network. Thus, these three different types of networks jointly cover a wide range of network density and size. See Table 1 for more details.

Given a network, we simulate data with the following data-generating mechanism: For units  $i = 1, \dots, n$ ,

- (1) Unobserved confounder with network dependence:  $U_i = \sum_{s \geq 0} \zeta^s \sum_{j \in \mathcal{N}(i; s)} \tilde{U}_j / |\mathcal{N}(i; s)|$  where  $\zeta = 0.8$  and  $\tilde{U}_j \stackrel{\text{i.i.d.}}{\sim} \text{Normal}(0, 1)$ . This data generating process for a network-dependent variable follows a simulation setup of Kojevnikov *et al.* (2020).
- (2) Observed covariates with network dependence:  $\mathbf{X}_i = (X_{i1}, X_{i2}, X_{i3})$  where, for  $k \in \{1, 2, 3\}$ ,  $X_{ik} = \sum_{s \geq 0} \zeta^s \sum_{j \in \mathcal{N}(i; s)} \tilde{X}_{jk} / |\mathcal{N}(i; s)|$ ,  $\zeta = 0.8$ , and  $\tilde{X}_{jk} \stackrel{\text{i.i.d.}}{\sim} \text{Normal}(0, 1)$ .
- (3) Observed auxiliary variable:  $C_i = U_i + \beta_c^\top \mathbf{X}_i + \epsilon_{i0}$  where  $\epsilon_{i0} \stackrel{\text{i.i.d.}}{\sim} \text{Normal}(0, 1)$  and  $\beta_c = (0.05, 0.05, 0.05)$ .
- (4) Focal behavior at the baseline:  $Y_{i1} = U_i + 0.05C_i + \beta_1^\top \mathbf{X}_i + \epsilon_{i1}$  where  $\epsilon_{i1} \stackrel{\text{i.i.d.}}{\sim} \text{Normal}(0, 1)$  and  $\beta_1 = (-1, -1, -1)$ .
- (5) Focal behavior at the follow-up:  $Y_{i2} = \tau A_i + 0.2Y_{i1} + 3U_i + 0.05C_i + \beta_2^\top \mathbf{X}_i + \epsilon_{i2}$  where  $\epsilon_{i2} \stackrel{\text{i.i.d.}}{\sim} \text{Normal}(0, 1)$ , and  $\beta_2 = (-1, -1, -1)$ . The treatment variable  $A_i$  is defined as  $A_i = \sum_{j \in \mathcal{N}(i; 1)} Y_{j1} / |\mathcal{N}(i; 1)|$ .

The above models imply that the ACPE is  $\tau$ , which we set to be 0.3. We can use  $W_i = C_i$  as the NCO, and  $Z_i = \sum_{j \in \mathcal{N}(i; 1)} C_j / |\mathcal{N}(i; 1)|$  as the NCE. Under this setup, the linear confounding bridge function,  $h(W_i, A_i, \mathbf{X}_i; \gamma) = \gamma_\alpha + \tau A_i + \gamma_W W_i + \gamma_X^\top \mathbf{X}_i$ , satisfies Assumption 2.

We evaluate the performance of the proposed DNC estimator and the network HAC variance estimator. We evaluate two choices of bandwidth for the network HAC variance estimator. First, we use the bandwidth of 2, which we analytically derive based on Lemma 4. The required Assumption 4 holds in this simulation design. Second, we also use the default bandwidth  $b_n$  (equation (19)) suggested in Kojevnikov *et al.* (2020). We use the Parzen kernel function.<sup>5</sup>

For reference, we also report the ordinary least squares estimator where we regress  $Y_{i2}$  on the treatment variable  $A_i$  and a set of observed variables  $(Y_{i1}, C_i, X_{i1}, X_{i2}, X_{i3})$ . This estimator is consistent only under conditional ignorability, which is violated due to unmeasured network confounder  $U_i$  under this simulation setup. Thus, this OLS estimator quantifies the amount of network confounding that the DNC estimator has to correct for.

**Results.** We generate 2000 simulations and evaluate estimators in terms of absolute mean bias, standard error (computed as standard deviation of point estimates across simulations), root mean squared error (RMSE), and coverage of 95% confidence intervals based on the network HAC variance estimator. We standardize the first three quantities by the true ACPE to ease interpretation. Table 1 summarizes the results of the simulation study. The performance of

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<sup>5</sup> $\omega(x) = 1 - 6x^2 + 6|x|^3$  if  $0 \leq |x| \leq 1/2$ ,  $\omega(x) = 2(1 - |x|)^3$  if  $1/2 < |x| \leq 1$ , and  $\omega(x) = 0$  if  $1 < |x|$ .

Simulation Design					DNC					OLS		
Network	Sample Size	Average $ \mathcal{N}_n(i; 1) $	Average $ \mathcal{N}_n(i; 2) $	Density	Bias	Standard Error	RMSE	Coverage (Analytical)	Coverage (Default)	Bias	Standard Error	RMSE
SW-4	500	4.00	10.05	0.80	0.14	0.63	0.65	0.96	0.96	0.98	0.26	1.02
	1000	4.00	10.10	0.40	0.07	0.42	0.42	0.95	0.95	1.00	0.18	1.01
	2000	4.00	10.13	0.20	0.03	0.28	0.28	0.95	0.94	1.01	0.13	1.01
	4000	4.00	10.14	0.10	0.02	0.20	0.20	0.95	0.94	1.00	0.09	1.01
SW-8	500	8.00	35.93	1.60	0.25	1.23	1.26	0.96	0.96	0.88	0.35	0.95
	1000	8.00	36.64	0.80	0.11	0.58	0.59	0.96	0.96	0.88	0.25	0.91
	2000	8.00	37.01	0.40	0.05	0.38	0.39	0.94	0.94	0.90	0.17	0.92
	4000	8.00	37.17	0.20	0.02	0.26	0.26	0.95	0.95	0.89	0.12	0.90
Add Health	500	3.82	20.85	0.77	0.15	0.72	0.74	0.96	0.87	0.98	0.29	1.02
	1000	4.80	26.72	0.48	0.06	0.46	0.46	0.95	0.94	0.95	0.20	0.97
	2000	5.69	31.88	0.28	0.03	0.31	0.31	0.95	0.95	0.93	0.15	0.94
	4000	5.95	33.24	0.15	0.02	0.22	0.22	0.94	0.94	0.92	0.10	0.92

Table 1: Operating Characteristics of Estimators under Different Networks.

*Note:* We consider three different networks; the small world network model with the average degree of four (SW-4) and eight (SW-8), and the Add Health network. We report the average degree, the average number of the peers-of-peers, and the density for each network with each sample size. For the DNC estimator, we report the absolute mean bias, the standard error, the RMSE, and coverage of the 95% confidence intervals based on the analytical bandwidth and the default bandwidth. For reference, we also report the absolute mean bias, the standard error, and the RMSE of the OLS estimator. The absolute mean bias, the standard error, and the RMSE for both estimators are standardized by the true ACPE.

the OLS estimator for coverage is not shown because all are close to zero due to substantial unmeasured confounding.

Our proposed DNC estimator remained stable with relatively small bias across all scenarios, and the bias reduced as sample size increased. As expected, standard errors of the proposed DNC estimators were larger than the biased OLS estimators, but the RMSE of the DNC estimator was smaller due to smaller bias. Compared to the low-density small-world network (SW-4), bias, standard errors, and RMSE were larger in the high-density small-world network (SW-8). Results for the Add Health network fell somewhere in between. The coverage of 95% confidence intervals was close to the nominal level when the analytical bandwidth was chosen. While coverage with default bandwidth tended to under-cover slightly at smaller sample sizes in the Add Health network structure, it improved as sample size increased. They indicated that our proposed standard error estimation provided valid inference. These results confirmed our theoretical results in finite sample and demonstrated the advantages of the proposed DNC estimator.

## 5 Empirical Application: Causal Peer Effects in Education

We apply our method to Add Health data to evaluate causal peer effects of education outcomes in a friendship network. The study of causal peer effects in education has a long history in the social sciences, and many studies have shown moderate positive peer effects (e.g., Epple and Romano, 2011; Sacerdote, 2011). While some papers have used experimental or quasi-experimental methods where classmates or roommates are randomly assigned by schools, the vast majority of existing evidence comes from observational studies of causal peer effects. In the absence of randomization, researchers have adjusted for a variety of observed pre-treatment covariates and a host of fixed effects (e.g., fixed effects for schools or network components). However, such approaches rely on a conditional ignorability assumption, and assume away unmeasured latent homophily or contextual confounding. For example, students who have higher education performance might become friends with other high-performing students due to unobserved characteristics. In this case, strong association between one’s education outcome and her friends’ outcomes cannot be interpreted as the causal peer effect. Potential bias due to such unmeasured network confounding can undermine the validity of causal conclusions. To explore the possibility of unmeasured network confounding, we use the proposed double negative control approach to identify and estimate the ACPE.

### 5.1 Data

Add Health project collected survey data from students in grades 7–12 from a nationally representative sample of over 100 private and public schools in years 1994–1995 in the United States. There were two types of surveys conducted in years 1994–1995, both of which provide information about social and demographic characteristics of respondents, education level and occupation of their parents, and their friendship links (i.e., their best friends, up to five females and up to five males). The first type was an in-school survey, which were administered to students in schools from September 1994 until April 1995. It includes over 90,000 students across about 140 schools. Each school administration occurred on a single day within one 45- to 60-minute class period. The second type was an in-home survey, in which students answered a 90-minute in-home interview. It includes over 20,000 students. The in-home interview was conducted at least 90 days after the in-school survey, except for 10 students who we exclude from analysis. Our analysis focuses on 10,264 students who completed both in-school and in-home surveys and answered questions related to variables we use below. The in-school survey serves as the baseline time period and the in-home survey as the follow-up period.

We examine a network based on the friendship information collected in the in-school survey.

We define friendships as symmetric relationships: the pair of students  $i$  and  $j$  in the same school are coded as friends if either  $i$  lists  $j$  as a friend, or  $j$  lists  $i$  as a friend, or both. The average degree of the friendship network is 6.20.

## 5.2 Setup

We use the grade-point average (GPA) at follow-up period  $Y_{i2}$  as the outcome variable. GPA ranges between 1 and 4, and is computed based on the average grade-point of four subjects; English, Mathematics, History/Social Studies, and Science. The treatment variable is the average GPA of the network peers at baseline  $A_i = \sum_{j \in \mathcal{N}(i;1)} Y_{j1} / |\mathcal{N}(i;1)|$  where  $|\mathcal{N}(i;s)|$  denotes the number of the  $s$ -th order network peers of unit  $i$ .

Following Section 3.3, we consider two sets of negative controls. First, we consider focal behaviors of peers and those measured at baseline as negative controls. In particular, we use the GPA at baseline  $Y_{i1}$  as the NCO, i.e.,  $W_i = Y_{i1}$ . We then use the average GPA of peers-of-peers at baseline as the NCE, i.e.,  $Z_i = \sum_{j \in \mathcal{N}(i;2)} Y_{j1} / |\mathcal{N}(i;2)|$ . These are valid negative controls when the GPAs of students are recorded concurrently within schools. Second, we also use an auxiliary variable for negative controls. To make the assumption about the confounding bridge function (Assumption 2.3) more plausible, we again use GPA at baseline as the NCO, i.e.,  $W_i = Y_{i1}$ , while we use level of peers' headaches as the NCE. Specifically, we use the average level of headaches of peers as NCE, i.e.,  $\tilde{Z}_i = \sum_{j \in \mathcal{N}(i;1)} C_{j1} / |\mathcal{N}(i;1)|$  where  $C_{j1}$  captures level of headaches by student  $j$  measured at baseline. According to Cohen-Cole and Fletcher (2008), whether a student has headaches is a plausible negative control because it is unlikely (a) to causally affect whether students are friends to each other and (b) to causally affect headaches of other units.

In accordance with prior analyses of these data in the literature, we also include a series of observed pre-treatment covariates  $\mathbf{X}_i$ , including fixed effects for each school, the number of friends, age, gender, race, born in the US, health status, physical fitness, school attendance, student grade, motivation in education, school attachment, homework, self esteem, household size, living with mother, living with father, parental care, mother's education, father's education, whether both parents work, relationship with teachers, and relationship with friends.

Following Section 3.6, we use a linear DNC estimator. In particular, we specify a linear confounding bridge as,

$$h(W_i, A_i, \mathbf{X}_i; \gamma) = \gamma_\alpha + \gamma_A A_i + \gamma_W W_i + \gamma_X^\top \mathbf{X}_i,$$

where  $(A_i, W_i)$  are the treatment variable and the NCO, and  $\mathbf{X}_i$  represent observed pre-treatment covariates defined above, including fixed effects for schools.



Method	Estimate	Stand. Error	p-value	95% CI
OLS under conditional ignorability	0.176	0.015	0.000	(0.147, 0.206)
DNC Estimator with NCE $Z_i$ : the average GPA of $\mathcal{N}(i; 2)$	0.033	0.049	0.497	(-0.063, 0.129)
DNC Estimator with NCE $\tilde{Z}_i$ : the average level of headaches of $\mathcal{N}(i; 1)$	0.078	0.183	0.668	(-0.280, 0.437)

Table 2: Estimated Average Causal Peer Effects on GPA.

*Note:* We report point estimates, standard errors, p-values, and 95% confidence intervals for each method.

Our causal estimand is  $\gamma_A$ , which captures the causal effect on a given student’s GPA induced by one point increase in the average GPA of her peers. We use the proposed network HAC variance estimator to compute standard errors and 95% confidence intervals. We report standard errors based on the analytical choice of the bandwidth described in Section 3.6, while results for the default bandwidth selection are similar and therefore not reported.

We compare our proposed DNC estimator to the OLS regression estimator, which relies on the assumption of conditional ignorability where we adjust for pre-treatment outcome  $Y_{i1}$  and  $\mathbf{X}_i$ . In absence of unmeasured network confounding, we expect DNC estimates and OLS estimates to be comparable (i.e. within sampling variability). For the OLS estimator, we apply the network HAC variance estimator (Kojevnikov *et al.*, 2020) to residuals in order to make the comparison clear. We note that the estimated standard errors for the OLS are valid only when there is no unmeasured network confounding.

### 5.3 Results

Table 2 reports estimates of the ACPE, standard errors, p-values, and 95% confidence intervals for each method. The OLS estimate suggests that the estimated ACPE is as large as 0.176 and statistically significant. However, our proposed DNC estimator indicates that there may be a large amount of unmeasured network confounding operating in this network which cannot be accounted for by a standard regression analysis, despite having accounted for a large number of pre-treatment covariates. The DNC estimate based on NCE  $Z_i$  is 0.033, is less than 20% of the OLS estimate, and is not statistically significant. The DNC estimate based on NCE  $\tilde{Z}_i$  shows a similar pattern: a point estimate is 0.078 and is not statistically significant. The standard error based on the second NCE  $\tilde{Z}_i$  is larger than the one based on the first NCE partly because the association between the NCO and NCE is weaker for the second NCE (Miao *et al.*,

2018a). These results show that the OLS estimator under conditional ignorability can suffer from more than 100% bias, which is consistent with previous validation studies on peer effects in the literature (e.g., Eckles and Bakshy, 2017).

## 6 Extension: Higher-order Peer Effects

Following standard causal peer effect literature, we have focused on the causal effect from peers as the causal estimand of primary interest (the ACPE defined in equation (7)). It is important to emphasize that all results in Section 3 do not rule out causal effects from higher-order peers (e.g., peers-of-peers). If they exist, one can simply adjust for focal behaviors of higher-order peers as observed pre-treatment covariates  $\mathbf{X}_i$ . We have considered such higher-order peer effects as nuisance when studying identification and estimation of the ACPE. In this section, we clarify that the proposed double negative control approach can also be used for identification and estimation of higher-order causal peer effects as well.

The study of such higher-order peer effects can be important for several reasons. First, in some applications, focal behaviors might be directly affected by higher-order peers even if peers might not change their behaviors. For example, information can diffuse from higher-order peers even if there is no behavioral change among peers. Second, estimation of higher-order peer effects can account for some forms of misspecification of underlying networks. It is possible that observed network and time might not perfectly match the underlying process through which units causally affect peers. For example, it is possible that units affect peers faster, and units can affect their peers-of-peers within one observed time interval. Additionally, the observed network might miss some ties between units, and thus, two units with the observed shortest distance of two might in fact be connected directly in the underlying true network. In such cases, we want to estimate causal effects from peers and peers-of-peers jointly.

One can explicitly include focal behaviors of higher-order peers into the potential outcome. Suppose we are interested in causal effects from all units within network distance  $s^\dagger$ . We define a vector of the treatment variable  $\tilde{A}_i = (A_{i1}, \dots, A_{is^\dagger})$  where  $A_{is} = \phi(\{Y_{j1} : j \in \mathcal{N}(i; s)\}) \in \mathbb{R}$ ,  $s \in \{1, \dots, s^\dagger\}$ , and function  $\phi$  is specified by a researcher based on subject matter knowledge. When  $s^\dagger = 1$ , this setup reduces to the one in Section 3. The potential outcome  $Y_{i2}(\tilde{a})$  is defined as the outcome that would realize when the treatment vector is set to  $\tilde{A}_i = \tilde{a}$ . We can then define the higher-order ACPE as

$$\tau(\tilde{a}, \tilde{a}') := \frac{1}{n} \sum_{i=1}^n \mathbb{E} \{Y_{i2}(\tilde{a}) - Y_{i2}(\tilde{a}')\} \quad (23)$$

where  $\tilde{a}, \tilde{a}' \in \tilde{\mathcal{A}}$  where  $\tilde{\mathcal{A}}$  is the support of  $\tilde{A}$ . For example,  $\tau((a_1, a_2), (a_1, a'_2))$  captures the second-order peer effect by fixing the treatment value of peers and changing the treatment

value of peers-of-peers. Importantly, while this setup considers up to the  $s^\dagger$ -th order peer effects as the causal estimand, this does not assume the absence of causal effects from peers at distance more than  $s^\dagger$ . We only view them as nuisance.

We can straightforwardly generalize Assumption 2 and Theorem 2 to this setting of higher-order peer effects by replacing  $A_i$  with  $\tilde{A}_i$ . The selection of negative controls can also proceed in similar fashion. A plausible candidate is again an auxiliary variable  $C_i$  that (a) does not affect network relationships and (b) does not affect variables of other units. For example, even if we add the second-order peer effects to Figure 3.(i) (i.e., a causal arrow from  $Y_{41}$  to  $Y_{22}$ ), the original choice of negative controls —  $C_2$  as the NCO and  $\{C_1, C_3, C_4\}$  as the NCEs — remains valid.

Another candidate for negative controls is the focal behavior itself. For example, if one were to add the second-order peer effects to Figure 3.(ii) (i.e., an causal arrow from  $Y_{41}$  to  $Y_{22}$ ), the original choice of NCO  $Y_{21}$  would remain valid, while the original choice of NCEs  $\{Y_{41}, Y_{42}\}$  would no longer be valid. If all third-order peer effects are absent, focal behaviors of third-order peers would be a plausible candidate for the NCEs. In summary, while the specific choice of negative controls need to be adjusted when examining higher-order ACPE, the two primary ways of selecting negative controls we discussed in Section 3 continue to be useful.

Finally, estimation and inference can proceed as in Theorems 3 and 4 can be extended by replacing  $A_i$  with  $\tilde{A}_i$  in the definition of  $\mathbf{L}_{n,i}$ .

## 7 Concluding Remarks

In this article, we have developed the double negative control approach to identification and estimation of causal peer effects. In contrast to existing literature, we take into account both unmeasured network confounding and network dependence of observations. We discuss two general approaches for selecting negative controls from network data in practice. One is based on an auxiliary variable that (a) does not affect network relationships and (b) does not affect variables of other units. The other plausible candidates for negative controls are focal behaviors of peers and those measured at baseline. We then provide a GMM estimator for the average causal peer effect and establish its consistency and asymptotic normality under conditions of  $\psi$ -network dependence. We also derive the network HAC variance estimator, with which researchers can construct asymptotic confidence intervals.

Our findings have established a theoretical basis for future research on nonparametric estimation of causal peer effects with double negative controls for unmeasured network confounding. This will be able to extend previous studies that consider double negative control adjustment of unmeasured confounding in i.i.d. or panel data settings (Deaner, 2018; Shi *et al.*, 2020; Cui

*et al.*, 2020; Tchetgen Tchetgen *et al.*, 2020a; Ghassami *et al.*, 2021) and methods that examine network effects without unmeasured network confounding (van der Laan, 2014; Ogburn *et al.*, 2017; Forastiere *et al.*, 2020; Ogburn *et al.*, 2020; Tchetgen Tchetgen *et al.*, 2020b). Another interesting open question is identification and estimation of causal peer effects in complex longitudinal studies with time-varying treatments (e.g., Robins *et al.*, 2000; Tchetgen Tchetgen *et al.*, 2020a).

## Supplementary Materials

The supplemental materials contain proofs of all the results described in the main text, as well as auxiliary results and proofs used to demonstrate the desired theoretical properties. We also provide additional simulation results.

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Supplementary Materials for  
 “Identification and Estimation of Causal Peer Effects Using  
 Double Negative Controls for Unmeasured Network Confounding”

## A Identification

### A.1 Proof of Lemma 1

In this proof, to make a discussion general, we use  $Y$  to denote the outcome and use  $A$  to denote the treatment instead of using  $(Y_{12}, Y_{21})$  and  $(Y_{i2}, A_i)$ , which we use in Section 2 and Section 3, respectively. To provide rigorous discussions on the existence of a solution to a Fredholm integral equation of the first kind, we rely on Picard’s theorem (Kress, 1989, Theorem 15.18).

**Lemma 5 (Picard’s theorem (Kress, 1989, Theorem 15.18))** *Given Hilbert spaces  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , let  $K : \mathcal{S}_1 \rightarrow \mathcal{S}_2$  be a compact operator with singular system  $(\nu_p, v_p, \kappa_p)_{p=1}^{+\infty}$ . Define its adjoint to be  $K^* : \mathcal{S}_2 \rightarrow \mathcal{S}_1$ . Then, for  $h \in \mathcal{S}_1$  and  $\tilde{h} \in \mathcal{S}_2$ , there exists a solution to a Fredholm integral equation of the first kind  $Kh = \tilde{h}$  if and only if (1)  $\tilde{h} \in \text{Null}(K^*)^\perp$  and (2)  $\sum_{p=1}^{+\infty} \frac{1}{\nu_p^2} |\langle \tilde{h}, \kappa_p \rangle|^2 < +\infty$ , where the inner product is defined for a Hilbert space  $\mathcal{S}_2$ ,  $\text{Null}(K^*) = \{\tilde{h} : K^*\tilde{h} = 0\}$  is the null space of  $K^*$ , and  $\perp$  represents the orthogonal complement to a subset.*

To apply this Picard’s theorem, we need to provide some additional notations. We use  $F$  and  $dF$  to denote the cumulative distribution function and the Radon-Nikodym derivative of  $F$ . We define  $L^2\{F(t)\}$  to be the space of all square integrable functions of  $t$  with respect to a cumulative distribution function  $F(t)$ , which is a Hilbert space with the inner product

$$\langle h_1, h_2 \rangle := \int_{-\infty}^{+\infty} h_1(t)h_2(t)dF(t) \quad \text{for all } h_1, h_2 \in L^2\{F(t)\}.$$

We define a kernel

$$K(w, u, a, \mathbf{x}) = \frac{dF(w, u \mid a, \mathbf{x})}{dF(w \mid a, \mathbf{x})dF(u \mid a, \mathbf{x})}.$$

We then define the linear operators  $K_{a, \mathbf{x}} : L^2\{F(w \mid a, \mathbf{x})\} \rightarrow L^2\{F(u \mid a, \mathbf{x})\}$  by

$$K_{a, \mathbf{x}}h = \int_{-\infty}^{+\infty} K(w, u, a, \mathbf{x})h(w)dF(w \mid a, \mathbf{x}) = \mathbb{E}\{h(w) \mid u, a, \mathbf{x}\}$$

for  $h \in L^2\{F(w | a, \mathbf{x})\}$ . The adjoint of this linear operator  $K_{a,\mathbf{x}}^* : L^2\{F(u | a, \mathbf{x})\} \rightarrow L^2\{F(w | a, \mathbf{x})\}$  is given by

$$K_{a,\mathbf{x}}^* \tilde{h} = \int_{-\infty}^{+\infty} K(w, u, a, \mathbf{x}) \tilde{h}(u) dF(u | a, \mathbf{x}) = \mathbb{E}\{\tilde{h}(u) | w, a, \mathbf{x}\}$$

for  $\tilde{h} \in L^2\{F(u | a, \mathbf{x})\}$ .

We first assume that  $W$  is relevant for  $U$ .

**Assumption 5 (Relevance of  $W$  for  $U$ )** *For any square integrable function  $f$  and any  $a$  and  $\mathbf{x}$ , if  $\mathbb{E}\{f(U) | W = w, A = a, \mathbf{X} = \mathbf{x}\} = 0$  for almost all  $w$ , then  $f(U) = 0$  almost surely.*

This is formally known as the completeness condition, and can be interpreted similarly to Assumption 1.4. We also introduce regularity conditions related to the singular value decomposition.

**Assumption 6 (Regularity Conditions)**

$$\int_{-\infty}^{+\infty} dF(u | w, a, \mathbf{x}) dF(w | u, a, \mathbf{x}) dw du < +\infty \quad (\text{A.1})$$

$$\int_{-\infty}^{+\infty} \mathbb{E}(Y | a, u, \mathbf{x})^2 dF(u | a, \mathbf{x}) du < +\infty \quad (\text{A.2})$$

$$\sum_{p=1}^{+\infty} \frac{1}{\nu_{a,\mathbf{x},p}^2} |\langle \mathbb{E}(Y | a, u, \mathbf{x}), \kappa_{a,\mathbf{x},p} \rangle|^2 < +\infty \quad (\text{A.3})$$

where  $\nu_{a,\mathbf{x},p}$  is the  $p$ -th singular value of  $K_{a,\mathbf{x}}$ , and  $\kappa_{a,\mathbf{x},p} \in L^2\{F(u | a, \mathbf{x})\}$  is an orthogonal sequence.

Under Assumptions 5 and 6, we prove the existence of a solution to the following Fredholm integral equation of the first kind.

$$\mathbb{E}(Y | A = a, U = u, \mathbf{X} = \mathbf{x}) = \mathbb{E}\{h(W, a, \mathbf{x}) | A = a, U = u, \mathbf{X} = \mathbf{x}\}. \quad (\text{A.4})$$

First, we can re-write equation (A.4) as follows using the notations introduced above.

$$K_{a,\mathbf{x}} h = \mathbb{E}(Y | A = a, U = u, \mathbf{X} = \mathbf{x}). \quad (\text{A.5})$$

Therefore, to use Picard's theorem, we need to prove (i)  $K_{a,\mathbf{x}}$  is a compact operator, (ii)  $\mathbb{E}(Y | A = a, U = u, \mathbf{X} = \mathbf{x}) \in L^2\{F(u | a, \mathbf{x})\}$ , (iii)  $\mathbb{E}(Y | A = a, U = u, \mathbf{X} = \mathbf{x}) \in \text{Null}(K_{a,\mathbf{x}}^*)^\perp$ , and

(iv)  $\sum_{p=1}^{+\infty} \frac{1}{\nu_{a,\mathbf{x},p}^2} |\langle \mathbb{E}(Y | A = a, U = u, \mathbf{X} = \mathbf{x}), \kappa_{a,\mathbf{x},p} \rangle|^2 < +\infty$ , where  $\nu_{a,\mathbf{x},p}$  is the  $p$ -th singular value of  $K_{a,\mathbf{x}}$ , and  $\kappa_{a,\mathbf{x},p} \in L^2\{F(u | a, \mathbf{x})\}$  is an orthogonal sequence.

Proof of (i): We note that  $K_{a,\mathbf{x}}$  and  $K_{a,\mathbf{x}}^*$  are compact operators under equation (A.1) (Carrasco *et al.*, 2007, Example 2.3 on page 5659). Therefore, there exists a singular system  $(\nu_{a,\mathbf{x},p}, v_{a,\mathbf{x},p}, \kappa_{a,\mathbf{x},p})$  of  $K_{a,\mathbf{x}}$  according to Kress (1989, Theorem 15.16) where  $\nu_{a,\mathbf{x},p}$  is the  $p$ -th singular value of  $K_{a,\mathbf{x}}$ , and  $v_{a,\mathbf{x},p} \in L^2\{F(w | a, \mathbf{x})\}$  and  $\kappa_{a,\mathbf{x},p} \in L^2\{F(u | a, \mathbf{x})\}$  are orthogonal sequences.

Proof of (ii): Under equation (A.2), we have  $\mathbb{E}(Y | a, u, \mathbf{x}) \in L^2\{F(u | a, \mathbf{x})\}$ .

Proof of (iii): We show that  $\text{Null}(K_{a,\mathbf{x}}^*)^\perp = L^2\{F(u | a, \mathbf{x})\}$ . For any  $\tilde{h} \in \text{Null}(K_{a,\mathbf{x}}^*)$ , we have  $K_{a,\mathbf{x}}^* \tilde{h} = \mathbb{E}\{\tilde{h}(u) | w, a, \mathbf{x}\} = 0$  almost surely by the definition of the null space. Under Assumption 5 (Relevance of  $W$  for  $U$ ), we have  $\tilde{h}(u) = 0$  almost surely. Therefore,  $\text{Null}(K_{a,\mathbf{x}}^*)^\perp = L^2\{F(u | a, \mathbf{x})\}$ . Based on (ii), we have  $\mathbb{E}(Y | a, u, \mathbf{x}) \in L^2\{F(u | a, \mathbf{x})\}$  under equation (A.2), and therefore,  $\mathbb{E}(Y | a, u, \mathbf{x}) \in \text{Null}(K_{a,\mathbf{x}}^*)^\perp$ .

Proof of (iv): Finally, this key condition for Picard's theorem is directly implied by equation (A.3), which completes the proof.  $\square$

## A.2 Details on Completeness Conditions

In this section, to make discussions simpler, we only focus on two random variables  $W$  and  $Z$ . We say that  $Z$  is complete with respect to  $W$  if  $\forall f(W) \in L^2\{F(W)\}$ ,

$$\mathbb{E}\{f(W) | Z\} = 0 \text{ almost surely} \implies f(W) = 0 \text{ almost surely.} \quad (\text{A.6})$$

This completeness condition, also known as  $L^2$ -completeness, requires that the conditional expectation projection operator  $K : L^2\{F(W)\} \rightarrow L^2\{F(Z)\}$  be injective (i.e.,  $\text{Null}(K) = \{0\}$ ). Intuitively, this means that no information has been lost through projection of  $W$  on  $Z$ . A necessary and sufficient condition of completeness is given by the following lemma.

**Lemma 6 (Severini and Tripathi (2006); Andrews (2017))**  *$Z$  is complete with respect to  $W$  if and only if every non-constant random variable  $\lambda(W) \in L^2\{F(W)\}$  is correlated with some random variable  $\tilde{\lambda}(Z) \in L^2\{F(Z)\}$ .*

This formally captures the notion that completeness ensures that there is no loss of information through projection of  $W$  on  $Z$ .

As explained in Section 2.3, the completeness condition has been long used in statistics and econometrics. Originally in statistics, Lehmann and Scheffé (2012a,b) introduced the concept of completeness and used it to define estimators with minimal risk within unbiased estimators. They defined completeness as  $\mathbb{E}_\theta(f(V)) = 0$  for any  $\theta \in \Theta$  implying  $f(V) = 0$  a.s. with respect to some parameter space  $\Theta$  parameterizing the distribution space. Shao (2003) defined completeness with respect to a family of distributions, i.e.,  $\mathbb{E}_P(f(V)) = 0$  for any  $P \in \mathcal{P}$  implying  $f(V) = 0$  a.s. with respect to some family of  $\mathcal{P}$ . In our definition of the completeness (Assumption 1.4 and Assumption 2.4), we set  $\mathcal{P}$  to be the conditional distribution. If we define a family of distributions to be  $\mathcal{P} = \{F(W | Z) : Z \in \mathcal{Z}\}$  of random variable  $W$ , the connection between our definition of completeness and the traditional completeness condition given in Lehmann and Scheffé (2012a,b) becomes clear. In particular, we say that a family of distributions  $\mathcal{P} = \{F(W | Z) : Z \in \mathcal{Z}\}$  of random variable  $W$  is complete with respect to  $Z$  if  $\forall f(W) \in L^2\{F(W)\}$ ,  $\mathbb{E}_{F(W|Z)}\{f(W)\} = \mathbb{E}\{f(W) | Z\} = 0$  for almost all  $Z$  implies that  $f(W) = 0$  almost surely. This is equivalent to our definition given in equation (A.6).

Recently, completeness conditions have been extensively applied in the econometrics literature to obtain identification for a variety of nonparametric and semi-parametric models, most famously, in nonparametric models with instrumental variables (e.g., Ai and Chen, 2003; Newey and Powell, 2003; Chernozhukov *et al.*, 2007; Darolles *et al.*, 2011). Other examples include measurement error models (e.g., Hu and Schennach, 2008) and panel or dynamic models (e.g., Hu and Shum, 2012; Freyberger, 2018).

Finally, as in our paper, completeness conditions have been essential in the literature of negative controls and proximal causal learning (Tchetgen Tchetgen *et al.*, 2020a). Miao *et al.* (2018b) make two completeness conditions (a) the completeness of  $W$  with respect to  $Z$ , (b) the completeness of  $Z$  with respect to  $U$  (see Conditions 2 and 3 in their paper). Deaner (2018); Shi *et al.* (2020); Kallus *et al.* (2021) make alternative two completeness conditions (a) the completeness of  $W$  with respect to  $U$ , (b) the completeness of  $Z$  with respect to  $U$  (see Assumption 3 in Deaner (2018), Assumption 4 in Shi *et al.* (2020), and Example 6 in Kallus *et al.* (2021)). Miao *et al.* (2018a) make one completeness condition (the completeness of  $Z$  with respect to  $W$ ; see Assumption 5 in their paper) along with the assumption of the existence of an outcome confounding bridge function, which can be justified by another completeness condition

(the completeness of  $W$  with respect to  $U$ ).

In Sections 2 and 3, we followed Miao *et al.* (2018a) and made Assumptions 1.3 and 1.4 and Assumptions 2.3 and 2.4, respectively. We prove nonparametric identification of the ACPE under those assumptions in Section A.5 below. We also prove nonparametric identification of the ACPE under an alternative set of completeness conditions in Section A.6 as well.

### A.3 Proof of Lemma 2

First, equation (9) implies that

$$C_i \perp\!\!\!\perp (\{C_j : j \neq i\}, A_i) \mid U_i, \mathbf{X}_i, \quad (\text{A.7})$$

$$\implies W_i \perp\!\!\!\perp A_i \mid U_i, \mathbf{X}_i, \quad (\text{A.8})$$

as we define  $W_i = C_i$  and  $A_i = \phi(\{Y_{j1} : j \in \mathcal{N}(i; 1)\}) \in \mathbb{R}$ . Then, equation (A.7) also implies that

$$C_i \perp\!\!\!\perp \{C_j : j \neq i\} \mid A_i, U_i, \mathbf{X}_i,$$

$$\implies W_i \perp\!\!\!\perp \mathbf{Z}_i \mid A_i, U_i, \mathbf{X}_i, \quad (\text{A.9})$$

as we define  $W_i = C_i$  and  $\mathbf{Z}_i = \{C_j : j \neq i\}$ . Finally, equation (10) implies that

$$Y_{i2} \perp\!\!\!\perp \{C_j : j \neq i\} \mid A_i, U_i, \mathbf{X}_i,$$

$$\implies Y_{i2} \perp\!\!\!\perp \mathbf{Z}_i \mid A_i, U_i, \mathbf{X}_i, \quad (\text{A.10})$$

where  $\mathbf{Z}_i = \{C_j : j \neq i\}$ . Therefore, equations (A.8)–(A.10) are equivalent to Assumption 2.2, which completes the proof.  $\square$

### A.4 Proof of Lemma 3

First, equation (11) implies that

$$Y_{i1} \perp\!\!\!\perp (A_i, \{Y_{j1} : j \in \mathcal{N}(i; s), s \geq 2\}) \mid U_i, \mathbf{X}_i, \quad (\text{A.11})$$

$$\implies W_i \perp\!\!\!\perp A_i \mid U_i, \mathbf{X}_i, \quad (\text{A.12})$$

as we define  $W_i = Y_{i1}$ . Then, equation (A.11) also implies that

$$Y_{i1} \perp\!\!\!\perp \{Y_{j1} : j \in \mathcal{N}(i; s), s \geq 2\} \mid A_i, U_i, \mathbf{X}_i,$$

$$\implies W_i \perp\!\!\!\perp \mathbf{Z}_i \mid A_i, U_i, \mathbf{X}_i, \quad (\text{A.13})$$



as we define  $W_i = Y_{i1}$  and  $\mathbf{Z}_i = \{Y_{j1} : j \in \mathcal{N}(i; s), s \geq \tilde{s}\}$  where  $\tilde{s} \geq 2$ .

Finally, equation (12) states that

$$\begin{aligned} Y_{i2} &\perp\!\!\!\perp \{Y_{j1} : j \in \mathcal{N}(i; s), s \geq \tilde{s}\} \mid A_i, U_i, \mathbf{X}_i, \\ \implies Y_{i2} &\perp\!\!\!\perp \mathbf{Z}_i \mid A_i, U_i, \mathbf{X}_i, \end{aligned} \tag{A.14}$$

where  $\mathbf{Z}_i = \{Y_{j1} : j \in \mathcal{N}(i; s), s \geq \tilde{s}\}$ . Therefore, equations (A.12)–(A.14) are equivalent to Assumption 2.2, which completes the proof.  $\square$

## A.5 Proof of Theorem 2

Here, we prove identification of  $\mathbb{E}\{Y_{i2}(a)\}$  for  $a \in \mathcal{A}$  and a given unit  $i \in N_n$ , which is sufficient for proving identification of the ACPE. The proof of Theorem 1 is a special case of the proof we provide below.

This proof adopts the proof by Miao *et al.* (2018a) to our network setting. First, we prove that the mean potential outcomes can be identified as the mean of the outcome confounding bridge function.

$$\mathbb{E}\{Y_{i2}(a)\} = \mathbb{E}\{h(W_i, a, \mathbf{X}_i)\}.$$

**Proof:** Under Assumption 2.1,

$$\begin{aligned} \int \mathbb{E}(Y_{i2} \mid A_i = a, U_i = u, \mathbf{X}_i = \mathbf{x}) dF(U_i = u, \mathbf{X}_i = \mathbf{x}) &= \int \mathbb{E}\{Y_{i2}(a) \mid U_i = u, \mathbf{X}_i = \mathbf{x}\} dF(U_i = u, \mathbf{X}_i = \mathbf{x}) \\ &= \mathbb{E}\{Y_{i2}(a)\}. \end{aligned}$$

Under Assumption 2.2,

$$\begin{aligned} &\int \mathbb{E}\{h(W_i, a, \mathbf{X}_i) \mid A_i = a, U_i = u, \mathbf{X}_i = \mathbf{x}\} dF(U_i = u, \mathbf{X}_i = \mathbf{x}) \\ &= \int \mathbb{E}\{h(W_i, a, \mathbf{X}_i) \mid U_i = u, \mathbf{X}_i = \mathbf{x}\} dF(U_i = u, \mathbf{X}_i = \mathbf{x}) \\ &= \mathbb{E}\{h(W_i, a, \mathbf{X}_i)\}. \end{aligned}$$

Under Assumption 2.3,  $\mathbb{E}(Y_{i2} \mid A_i = a, U_i = u, \mathbf{X}_i = \mathbf{x}) = \mathbb{E}\{h(W_i, a, \mathbf{X}_i) \mid A_i = a, U_i = u, \mathbf{X}_i = \mathbf{x}\}$ , and therefore,

$$\mathbb{E}\{Y_{i2}(a)\} = \mathbb{E}\{h(W_i, a, \mathbf{X}_i)\},$$

which completes the proof.  $\square$

Next, we prove that the confounding bridge function is identified as follows.

$$\mathbb{E}(Y_{i2} \mid \mathbf{Z}_i, A_i, \mathbf{X}_i) = \mathbb{E}\{h(W_i, A_i, \mathbf{X}_i) \mid \mathbf{Z}_i, A_i, \mathbf{X}_i\}. \tag{A.15}$$

**Proof:** Under Assumption 2.2,

$$\begin{aligned}
& \int \mathbb{E}(Y_{i2} \mid A_i, U_i = u, \mathbf{X}_i) dF(U_i = u \mid \mathbf{Z}_i, A_i, \mathbf{X}_i) \\
&= \int \mathbb{E}(Y_{i2} \mid \mathbf{Z}_i, A_i, U_i = u, \mathbf{X}_i) dF(U_i = u \mid \mathbf{Z}_i, A_i, \mathbf{X}_i) \\
&= \mathbb{E}(Y_{i2} \mid \mathbf{Z}_i, A_i, \mathbf{X}_i).
\end{aligned}$$

Under Assumption 2.2,

$$\begin{aligned}
& \int \mathbb{E}\{h(W_i, A_i, \mathbf{X}_i) \mid A_i, U_i = u, \mathbf{X}_i\} dF(U_i = u \mid \mathbf{Z}_i, A_i, \mathbf{X}_i) \\
&= \int \mathbb{E}\{h(W_i, A_i, \mathbf{X}_i) \mid \mathbf{Z}_i, A_i, U_i = u, \mathbf{X}_i\} dF(U_i = u \mid \mathbf{Z}_i, A_i, \mathbf{X}_i) \\
&= \mathbb{E}\{h(W_i, A_i, \mathbf{X}_i) \mid \mathbf{Z}_i, A_i, \mathbf{X}_i\}.
\end{aligned}$$

Under Assumption 2.3,  $\mathbb{E}(Y_{i2} \mid A_i, U_i, \mathbf{X}_i) = \mathbb{E}\{h(W_i, A_i, \mathbf{X}_i) \mid A_i, U_i, \mathbf{X}_i\}$ , and therefore,

$$\mathbb{E}(Y_{i2} \mid \mathbf{Z}_i, A_i, \mathbf{X}_i) = \mathbb{E}\{h(W_i, A_i, \mathbf{X}_i) \mid \mathbf{Z}_i, A_i, \mathbf{X}_i\}.$$

We finally demonstrate that the solution to equation (A.15) is unique and identifies the outcome confounding bridge function  $h$  under Assumption 2.4. Suppose there are two functions  $h(W_i, A_i, \mathbf{X}_i)$  and  $h'(W_i, A_i, \mathbf{X}_i)$  that satisfy equation (A.15). Then,

$$\mathbb{E}\{h(W_i, A_i, \mathbf{X}_i) - h'(W_i, A_i, \mathbf{X}_i) \mid \mathbf{Z}_i = \mathbf{z}, A_i = a, \mathbf{X}_i = \mathbf{x}\} = 0$$

for all  $a, \mathbf{x}$ , and almost all  $\mathbf{z}$ . Then, under Assumption 2.4,  $h(W_i, A_i, \mathbf{X}_i) = h'(W_i, A_i, \mathbf{X}_i)$  almost surely. Thus, the solution to equation (A.15) identifies the outcome confounding bridge function.  $\square$

## A.6 Identification of the ACPE under Alternative Assumptions

Here, we show that the same identification formula for the ACPE can be proven based on an alternative set of assumptions. The main difference is that we first define an outcome bridge function as a solution to the following Fredholm integral equation of the first kind.

$$\mathbb{E}(Y_{i2} \mid \mathbf{Z}_i, A_i, \mathbf{X}_i) = \mathbb{E}\{h^\dagger(W_i, A_i, \mathbf{X}_i) \mid \mathbf{Z}_i, A_i, \mathbf{X}_i\}. \quad (\text{A.16})$$

Then, we show, under some assumptions, this outcome bridge function satisfies

$$\mathbb{E}(Y_{i2} \mid A_i = a, U_i, \mathbf{X}_i) = \mathbb{E}\{h^\dagger(W_i, a, \mathbf{X}_i) \mid A_i = a, U_i, \mathbf{X}_i\}. \quad (\text{A.17})$$

This approach is opposite to the approach we used in the main paper and proved in Section A.5 where we defined an outcome bridge function as a solution to equation (A.17) and then showed that it satisfies equation (A.16). The approach used in this section is similar to the one used in Deaner (2018); Miao *et al.* (2018b); Shi *et al.* (2020).

We see below that this difference in the proof approaches lead to a different set of assumptions, while they both result in the same identification formula for the ACPE.

In particular, while we maintain Assumption 2.1 and Assumption 2.2, we replace Assumption 2.3 and Assumption 2.4 with two different assumptions below (Assumptions 7 and 8).

**Assumption 7 (Outcome Confounding Bridge  $h^\dagger$ )** *There exists some function  $h^\dagger(W_i, A_i, \mathbf{X}_i)$  such that for all  $a \in \mathcal{A}$ , and all  $i \in N_n$ ,*

$$\mathbb{E}(Y_{i2} \mid A_i = a, \mathbf{Z}_i = \mathbf{z}, \mathbf{X}_i) = \mathbb{E}\{h^\dagger(W_i, a, \mathbf{X}_i) \mid A_i = a, \mathbf{Z}_i = \mathbf{z}, \mathbf{X}_i\}. \quad (\text{A.18})$$

**Assumption 8 (Relevance of  $\mathbf{Z}$  for  $U$ )** *For any square integrable function  $f$  and for any  $a$  and  $\mathbf{x}$ , if  $\mathbb{E}\{f(U_i) \mid \mathbf{Z}_i = \mathbf{z}, A_i = a, \mathbf{X}_i = \mathbf{x}\} = 0$  for almost all  $\mathbf{z}$ , then  $f(U_i) = 0$  almost surely.*

**Theorem 5** *Under Assumptions 2.1, 2.2, 7 and 8, an outcome confounding bridge function  $h^\dagger$  (defined in equation (A.18)) satisfies the following equality for all  $a \in \mathcal{A}$ , and all  $i \in N_n$ ,*

$$\mathbb{E}(Y_{i2} \mid U_i, A_i = a, \mathbf{X}_i) = \mathbb{E}\{h^\dagger(W_i, a, \mathbf{X}_i) \mid U_i, A_i = a, \mathbf{X}_i\},$$

and, the ACPE is identified by

$$\tau(a, a') = \frac{1}{n} \sum_{i=1}^n \mathbb{E}\{h^\dagger(W_i, a, \mathbf{X}_i) - h^\dagger(W_i, a', \mathbf{X}_i)\}.$$

Finally, like Lemma 1, we can also prove Assumption 7 under a completeness condition and associated regularity conditions (Assumption 9 defined below in Section A.6.2).

**Lemma 7** *Under Assumptions 5 and 9, there exists a function  $h^\dagger(W_i, a, \mathbf{X}_i)$  such that for all  $a \in \mathcal{A}$  and all  $i \in N_n$ , equation (A.18) holds.*

### A.6.1 Proof of Theorem 5

First, we show that an outcome confounding bridge function  $h^\dagger$  defined in equation (A.18) satisfies the following equality.

$$\mathbb{E}\{h^\dagger(W_i, a, \mathbf{X}_i) \mid A_i = a, U_i = u, \mathbf{X}_i = \mathbf{x}\} = \mathbb{E}\{Y_{i2}(a) \mid A_i = a, U_i = u, \mathbf{X}_i = \mathbf{x}\} \quad (\text{A.19})$$

We have

$$\begin{aligned}
& \mathbb{E}\{h^\dagger(W_i, a, \mathbf{X}_i) \mid A_i = a, \mathbf{Z}_i = \mathbf{z}, \mathbf{X}_i = \mathbf{x}\} \\
&= \int \mathbb{E}\{h^\dagger(W_i, a, \mathbf{X}_i) \mid A = a, U_i = u, \mathbf{Z}_i = \mathbf{z}, \mathbf{X}_i = \mathbf{x}\} dF(U_i = u \mid A = a, \mathbf{Z}_i = \mathbf{z}, \mathbf{X}_i = \mathbf{x}) \\
&= \int \mathbb{E}\{h^\dagger(W_i, a, \mathbf{X}_i) \mid A = a, U_i = u, \mathbf{X}_i = \mathbf{x}\} dF(U_i = u \mid A = a, \mathbf{Z}_i = \mathbf{z}, \mathbf{X}_i = \mathbf{x}) \quad (\text{A.20})
\end{aligned}$$

where the first equality follows from iterated expectations, and the second from Assumption 2.2.

We also have

$$\begin{aligned}
& \mathbb{E}(Y_{i2} \mid A_i = a, \mathbf{Z}_i = \mathbf{z}, \mathbf{X}_i = \mathbf{x}) \\
&= \int \mathbb{E}(Y_{i2} \mid A_i = a, U_i = u, \mathbf{Z}_i = \mathbf{z}, \mathbf{X}_i = \mathbf{x}) dF(U_i = u \mid A_i = a, \mathbf{Z}_i = \mathbf{z}, \mathbf{X}_i = \mathbf{x}) \\
&= \int \mathbb{E}(Y_{i2} \mid A_i = a, U_i = u, \mathbf{X}_i = \mathbf{x}) dF(U_i = u \mid A_i = a, \mathbf{Z}_i = \mathbf{z}, \mathbf{X}_i = \mathbf{x}) \quad (\text{A.21})
\end{aligned}$$

where the first equality follows from iterated expectations, and the second from Assumption 2.2.

Under Assumption 7, we have

$$\begin{aligned}
& \mathbb{E}(Y_{i2} \mid A_i = a, \mathbf{Z}_i = \mathbf{z}, \mathbf{X}_i) = \mathbb{E}\{h^\dagger(W_i, a, \mathbf{X}_i) \mid A_i = a, \mathbf{Z}_i = \mathbf{z}, \mathbf{X}_i\} \\
&\iff \int \{\mathbb{E}(Y_{i2} \mid A_i = a, U_i = u, \mathbf{X}_i = \mathbf{x}) - \mathbb{E}\{h^\dagger(W_i, a, \mathbf{X}_i) \mid A_i = a, U_i = u, \mathbf{X}_i = \mathbf{x}\}\} \\
&\quad \times dF(U_i = u \mid A_i = a, \mathbf{Z}_i = \mathbf{z}, \mathbf{X}_i = \mathbf{x}) = 0 \\
&\implies \mathbb{E}(Y_{i2}(a) \mid A_i = a, U_i = u, \mathbf{X}_i = \mathbf{x}) = \mathbb{E}\{h^\dagger(W_i, a, \mathbf{X}_i) \mid A_i = a, U_i = u, \mathbf{X}_i = \mathbf{x}\}
\end{aligned}$$

where the first equivalence comes from equations (A.20) and (A.21), and the final line follows from Assumption 8 and the consistency of the potential outcomes.

Next, by using equation (A.19), we prove that

$$\mathbb{E}\{Y_{i2}(a)\} = \mathbb{E}\{h^\dagger(W_i, a, \mathbf{X}_i)\}.$$

Under Assumption 2.1, we have

$$\begin{aligned}
& \int \mathbb{E}(Y_{i2}(a) \mid A_i = a, U_i = u, \mathbf{X}_i = \mathbf{x}) dF(U_i = u, \mathbf{X}_i = \mathbf{x}) \\
&= \int \mathbb{E}\{Y_{i2}(a) \mid U_i = u, \mathbf{X}_i = \mathbf{x}\} dF(U_i = u, \mathbf{X}_i = \mathbf{x}) \\
&= \mathbb{E}\{Y_{i2}(a)\}.
\end{aligned}$$

Under Assumption 2.2, we have

$$\int \mathbb{E}\{h^\dagger(W_i, a, \mathbf{X}_i) \mid A_i = a, U_i = u, \mathbf{X}_i = \mathbf{x}\} dF(U_i = u, \mathbf{X}_i = \mathbf{x})$$

$$\begin{aligned}
&= \int \mathbb{E}\{h^\dagger(W_i, a, \mathbf{X}_i) \mid U_i = u, \mathbf{X}_i = \mathbf{x}\} dF(U_i = u, \mathbf{X}_i = \mathbf{x}) \\
&= \mathbb{E}\{h^\dagger(W_i, a, \mathbf{X}_i)\}.
\end{aligned}$$

Equation (A.19) states that  $\mathbb{E}(Y_{i2}(a) \mid A_i = a, U_i = u, \mathbf{X}_i = \mathbf{x}) = \mathbb{E}\{h^\dagger(W_i, a, \mathbf{X}_i) \mid A_i = a, U_i = u, \mathbf{X}_i = \mathbf{x}\}$ , and therefore,

$$\mathbb{E}\{Y_{i2}(a)\} = \mathbb{E}\{h^\dagger(W_i, a, \mathbf{X}_i)\},$$

which completes the proof.  $\square$

### A.6.2 Proof of Lemma 7

In this proof, to make a discussion general, we use  $Y$  to denote the outcome and use  $A$  to denote the treatment instead of using  $(Y_{i2}, A_i)$ , which we use in Section 3. To provide rigorous discussions on the existence of a solution to a Fredholm integral equation of the first kind, we keep using some notations introduced in Section A.1.

Using general notations, we re-state Lemma 7 as follows. Under Assumptions 5 and 9, there exists a function  $h(W, a, \mathbf{X})$  such that for all  $a \in \mathcal{A}$ , a solution to the following Fredholm integral equation of the first kind exists.

$$\mathbb{E}(Y \mid \mathbf{Z} = \mathbf{z}, A = a, \mathbf{X} = \mathbf{x}) = \mathbb{E}\{h^\dagger(W, a, \mathbf{x}) \mid \mathbf{Z} = \mathbf{z}, A = a, \mathbf{X} = \mathbf{x}\}. \quad (\text{A.22})$$

We also introduce regularity conditions related to the singular value decomposition.

#### Assumption 9 (Regularity Conditions II)

$$\int_{-\infty}^{+\infty} dF(\mathbf{z} \mid w, a, \mathbf{x}) dF(w \mid \mathbf{z}, a, \mathbf{x}) dw d\mathbf{z} < +\infty \quad (\text{A.23})$$

$$\int_{-\infty}^{+\infty} \mathbb{E}(Y \mid a, \mathbf{z}, \mathbf{x})^2 dF(\mathbf{z} \mid a, \mathbf{x}) d\mathbf{z} < +\infty \quad (\text{A.24})$$

$$\sum_{p=1}^{+\infty} \frac{1}{\tilde{\nu}_{a,\mathbf{x},p}^2} |\langle \mathbb{E}(Y \mid a, \mathbf{z}, \mathbf{x}), \tilde{\kappa}_{a,\mathbf{x},p} \rangle|^2 < +\infty \quad (\text{A.25})$$

where  $\tilde{\nu}_{a,\mathbf{x},p}$  is the  $p$ -th singular value of  $\tilde{K}_{a,\mathbf{x}}$ , and  $\tilde{\kappa}_{a,\mathbf{x},p} \in L^2\{F(\mathbf{z} \mid a, \mathbf{x})\}$  is an orthogonal sequence.

**Proof:** We start by defining a kernel

$$\tilde{K}(w, \mathbf{z}, a, \mathbf{x}) = \frac{dF(w, \mathbf{z} \mid a, \mathbf{x})}{dF(w \mid a, \mathbf{x}) dF(\mathbf{z} \mid a, \mathbf{x})}.$$

We then define the linear operators  $\tilde{K}_{a,\mathbf{x}} : L^2\{F(w | a, \mathbf{x})\} \rightarrow L^2\{F(\mathbf{z} | a, \mathbf{x})\}$  by

$$\tilde{K}_{a,\mathbf{x}}h = \int_{-\infty}^{+\infty} \tilde{K}(w, \mathbf{z}, a, \mathbf{x})h(w)dF(w | a, \mathbf{x}) = \mathbb{E}\{h(w) | \mathbf{z}, a, \mathbf{x}\}$$

for  $h \in L^2\{F(w | a, \mathbf{x})\}$ .

The adjoint of this linear operator  $\tilde{K}_{a,\mathbf{x}}^* : L^2\{F(\mathbf{z} | a, \mathbf{x})\} \rightarrow L^2\{F(w | a, \mathbf{x})\}$  is given by

$$\tilde{K}_{a,\mathbf{x}}^*\tilde{h} = \int_{-\infty}^{+\infty} \tilde{K}(w, \mathbf{z}, a, \mathbf{x})\tilde{h}(\mathbf{z})dF(\mathbf{z} | a, \mathbf{x}) = \mathbb{E}\{\tilde{h}(\mathbf{z}) | w, a, \mathbf{x}\}$$

for  $\tilde{h} \in L^2\{F(\mathbf{z} | a, \mathbf{x})\}$ .

Using the introduced notations, we can re-write equation (A.22) as follows using the notations introduced above.

$$\tilde{K}_{a,\mathbf{x}}h = \mathbb{E}(Y | A = a, \mathbf{Z} = \mathbf{z}, \mathbf{X} = \mathbf{x}). \quad (\text{A.26})$$

Therefore, to use Picard's theorem (Lemma 5), we need to prove (i)  $\tilde{K}_{a,\mathbf{x}}$  is a compact operator, (ii)  $\mathbb{E}(Y | A = a, \mathbf{Z} = \mathbf{z}, \mathbf{X} = \mathbf{x}) \in L^2\{F(\mathbf{z} | a, \mathbf{x})\}$ , (iii)  $\mathbb{E}(Y | A = a, \mathbf{Z} = \mathbf{z}, \mathbf{X} = \mathbf{x}) \in \text{Null}(\tilde{K}_{a,\mathbf{x}}^*)^\perp$ , and (iv)  $\sum_{p=1}^{+\infty} \frac{1}{\tilde{\nu}_{a,\mathbf{x},p}^2} |\langle \mathbb{E}(Y | A = a, \mathbf{Z} = \mathbf{z}, \mathbf{X} = \mathbf{x}), \tilde{\kappa}_{a,\mathbf{x},p} \rangle|^2 < +\infty$ , where  $\tilde{\nu}_{a,\mathbf{x},p}$  is the  $p$ -th singular value of  $\tilde{K}_{a,\mathbf{x}}$ , and  $\tilde{\kappa}_{a,\mathbf{x},p} \in L^2\{F(\mathbf{z} | a, \mathbf{x})\}$  is an orthogonal sequence.

Proof of (i): We note that  $\tilde{K}_{a,\mathbf{x}}$  and  $\tilde{K}_{a,\mathbf{x}}^*$  are compact operators under equation (A.23) (Carrasco *et al.*, 2007, Example 2.3 on page 5659). Therefore, there exists a singular system  $(\tilde{\nu}_{a,\mathbf{x},p}, \tilde{\nu}_{a,\mathbf{x},p}, \tilde{\kappa}_{a,\mathbf{x},p})$  of  $\tilde{K}_{a,\mathbf{x}}$  according to Kress (1989, Theorem 15.16) where  $\tilde{\nu}_{a,\mathbf{x},p}$  is the  $p$ -th singular value of  $\tilde{K}_{a,\mathbf{x}}$ , and  $\tilde{\nu}_{a,\mathbf{x},p} \in L^2\{F(w | a, \mathbf{x})\}$  and  $\tilde{\kappa}_{a,\mathbf{x},p} \in L^2\{F(\mathbf{z} | a, \mathbf{x})\}$  are orthogonal sequences.

Proof of (ii): Under equation (A.24), we have  $\mathbb{E}(Y | a, \mathbf{z}, \mathbf{x}) \in L^2\{F(\mathbf{z} | a, \mathbf{x})\}$ .

Proof of (iii): To show that  $\mathbb{E}(Y | A = a, \mathbf{Z} = \mathbf{z}, \mathbf{X} = \mathbf{x}) \in \text{Null}(\tilde{K}_{a,\mathbf{x}}^*)^\perp$ , we first define  $\tilde{h} \in \text{Null}(\tilde{K}_{a,\mathbf{x}}^*)$ . Then, we show below that, for any  $\tilde{h} \in \text{Null}(\tilde{K}_{a,\mathbf{x}}^*)$ ,

$$\langle \mathbb{E}(Y | A = a, \mathbf{Z} = \mathbf{z}, \mathbf{X} = \mathbf{x}), \tilde{h}(\mathbf{z}) \rangle = 0.$$

We begin by showing  $\mathbb{E}\{\tilde{h}(\mathbf{z}) | U, a, \mathbf{x}\} = 0$ . First, we have

$$\begin{aligned} \mathbb{E}\{\tilde{h}(\mathbf{z}) | w, a, \mathbf{x}\} &= \mathbb{E}\{\mathbb{E}\{\tilde{h}(\mathbf{z}) | U, w, a, \mathbf{x}\} | w, a, \mathbf{x}\} \\ &= \mathbb{E}\{\mathbb{E}\{\tilde{h}(\mathbf{z}) | U, a, \mathbf{x}\} | w, a, \mathbf{x}\} \end{aligned} \quad (\text{A.27})$$

where the first equality follows from iterated expectations, and the second from Assumption 2.2.

By definition of the null space, we have  $\tilde{K}_{a,\mathbf{x}}^*\tilde{h} = \mathbb{E}\{\tilde{h}(\mathbf{z}) | w, a, \mathbf{x}\} = 0$  almost surely. Therefore,

$$\mathbb{E}\{\tilde{h}(\mathbf{z}) | w, a, \mathbf{x}\} = 0 \iff \mathbb{E}\{\mathbb{E}\{\tilde{h}(\mathbf{z}) | U, a, \mathbf{x}\} | w, a, \mathbf{x}\} = 0$$

$$\implies \mathbb{E}\{\tilde{h}(\mathbf{z}) \mid U, a, \mathbf{x}\} = 0 \quad \text{almost surely.} \quad (\text{A.28})$$

where the first equivalence follows from equation (A.27), and the second line follows from the relevance of  $W$  for  $U$  (Assumption 5). Finally, we now show  $\langle \mathbb{E}(Y \mid A = a, \mathbf{Z} = \mathbf{z}, \mathbf{X} = \mathbf{x}), \tilde{h}(\mathbf{z}) \rangle = 0$ .

$$\begin{aligned} & \langle \mathbb{E}(Y \mid A = a, \mathbf{Z} = \mathbf{z}, \mathbf{X} = \mathbf{x}), \tilde{h}(\mathbf{z}) \rangle \\ := & \mathbb{E}\{\mathbb{E}(Y \mid A = a, \mathbf{Z} = \mathbf{z}, \mathbf{X} = \mathbf{x})\tilde{h}(\mathbf{z}) \mid A = a, \mathbf{X} = \mathbf{x}\} \\ = & \mathbb{E}\{\mathbb{E}\{\mathbb{E}(Y \mid U, A = a, \mathbf{Z} = \mathbf{z}, \mathbf{X} = \mathbf{x}) \mid A = a, \mathbf{Z} = \mathbf{z}, \mathbf{X} = \mathbf{x}\}\tilde{h}(\mathbf{z}) \mid A = a, \mathbf{X} = \mathbf{x}\} \\ = & \mathbb{E}\{\mathbb{E}\{\mathbb{E}(Y \mid U, A = a, \mathbf{X} = \mathbf{x}) \mid A = a, \mathbf{Z} = \mathbf{z}, \mathbf{X} = \mathbf{x}\}\tilde{h}(\mathbf{z}) \mid A = a, \mathbf{X} = \mathbf{x}\} \\ = & \mathbb{E}\{\mathbb{E}\{\mathbb{E}\{\mathbb{E}(Y \mid U, A = a, \mathbf{X} = \mathbf{x}) \mid A = a, \mathbf{Z} = \mathbf{z}, \mathbf{X} = \mathbf{x}\}\tilde{h}(\mathbf{z}) \mid U, A = a, \mathbf{X} = \mathbf{x}\} \mid A = a, \mathbf{X} = \mathbf{x}\} \\ = & \mathbb{E}\{\mathbb{E}(Y \mid U, A = a, \mathbf{X} = \mathbf{x})\mathbb{E}\{\tilde{h}(\mathbf{z}) \mid U, A = a, \mathbf{X} = \mathbf{x}\} \mid A = a, \mathbf{X} = \mathbf{x}\} \\ = & 0, \end{aligned}$$

where the first line follows from the definition of the inner product in a Hilbert space, the second from iterated expectations applied to  $\mathbb{E}(Y \mid A = a, \mathbf{Z} = \mathbf{z}, \mathbf{X} = \mathbf{x})$ , the third from Assumption 2.2, the fourth from iterated expectations, the fifth from conditioning on  $(U, A = a, \mathbf{X} = \mathbf{x})$ , and finally, the sixth follows from equation (A.28). Therefore, this shows that  $\mathbb{E}(Y \mid A = a, \mathbf{Z} = \mathbf{z}, \mathbf{X} = \mathbf{x}) \in \text{Null}(\tilde{K}_{a,\mathbf{x}}^*)^\perp$ .

Proof of (iv): Finally, this key condition for Picard's theorem is directly implied by equation (A.25), which completes the proof.  $\square$

## B Asymptotic Properties of the DNC Estimator

### B.1 Setup and Regularity Conditions

To derive asymptotic properties of our estimator, we assume the standard GMM regularity conditions (Hansen, 1982; Newey and McFadden, 1994).

The GMM regularity conditions:

- Parameter space  $\Theta$  is compact.
- $m(\mathbf{L}_{n,i}; \theta)$  is differentiable in  $\theta \in \Theta$  with probability one.
- $m(\mathbf{L}_{n,i}; \theta)$  and  $\frac{\partial}{\partial \theta} m(\mathbf{L}_{n,i}; \theta)$  are continuous at each  $\theta \in \Theta$  with probability one.
- $\mathbb{E}\{m(\mathbf{L}_{n,i}; \theta)\} = 0$  only when  $\theta = \theta_0$ , and  $\theta_0$  is in the interior of  $\Theta$ .

- $\mathbb{E}\{m(\mathbf{L}_{n,i}; \theta)\}$  and  $\mathbb{E}\{\frac{\partial}{\partial \theta} m(\mathbf{L}_{n,i}; \theta)\}$  are continuous in  $\theta$ .
- $\frac{1}{n} \sum_{i=1}^n m(\mathbf{L}_{n,i}; \theta)$  is stochastically equicontinuous on  $\Theta$ .
- $\frac{1}{n} \sum_{i=1}^n \frac{\partial}{\partial \theta} m(\mathbf{L}_{n,i}; \theta)$  is stochastically equicontinuous on  $\Theta$ .
- $M_0 = \frac{1}{n} \sum_{i=1}^n \mathbb{E}\{\frac{\partial}{\partial \theta} m(\mathbf{L}_{n,i}; \theta_0)\}$  is full rank.
- For  $p$  that satisfies Assumption 3,  $\sup_{n \geq 1} \max_{i \in N_n} \mathbb{E}\{|c^\top m(\mathbf{L}_{n,i}; \theta)|^p\} < \infty$  for any  $c$  with  $\|c\|_2 = \sqrt{c^\top c} = 1$  for all  $\theta \in \Theta$ .
- For  $p$  that satisfies Assumption 3,  $\sup_{n \geq 1} \max_{i \in N_n} \mathbb{E}\{|\tilde{c}^\top \frac{\partial}{\partial \theta} m(\mathbf{L}_{n,i}; \theta)|^p\} < \infty$  for any  $\tilde{c}$  with  $\|\tilde{c}\|_2 = \sqrt{\tilde{c}^\top \tilde{c}} = 1$  for all  $\theta \in \Theta$ .

We first define the GMM objective function:

$$\begin{aligned}
Q_n(\theta) &= \left\{ \frac{1}{n} \sum_{i=1}^n \mathbb{E}\{m(\mathbf{L}_{n,i}; \theta)\} \right\}^\top \Omega \left\{ \frac{1}{n} \sum_{i=1}^n \mathbb{E}\{m(\mathbf{L}_{n,i}; \theta)\} \right\}, \\
\widehat{Q}_n(\theta) &= \left\{ \frac{1}{n} \sum_{i=1}^n m(\mathbf{L}_{n,i}; \theta) \right\}^\top \Omega \left\{ \frac{1}{n} \sum_{i=1}^n m(\mathbf{L}_{n,i}; \theta) \right\}.
\end{aligned}$$

Then, the GMM estimator of  $\theta$  can be written as:

$$\widehat{\theta} = \operatorname{argmin}_{\theta \in \Theta} \widehat{Q}_n(\theta). \quad (\text{A.29})$$

## B.2 Proof of Theorem 3

Given that our DNC estimator  $\widehat{\tau}(a, a')$  for  $\tau(a, a')$  corresponds to the first element of  $\widehat{\theta}$ , we state theoretical properties in terms of  $\widehat{\theta}$ , which imply Theorem 3.

**Consistency.** We first want to show consistency of the GMM estimator:

$$\widehat{\theta} \xrightarrow{p} \theta_0.$$

**Proof:** Under Assumption 3, Proposition 3.1 by Kojevnikov *et al.* (2020) implies point-wise convergence of  $\frac{1}{n} \sum_{i=1}^n m(\mathbf{L}_{n,i}; \theta)$ . That is, for all  $\theta \in \Theta$ ,

$$\frac{1}{n} \sum_{i=1}^n \{m(\mathbf{L}_{n,i}; \theta) - \mathbb{E}\{m(\mathbf{L}_{n,i}; \theta)\}\} \xrightarrow{p} 0. \quad (\text{A.30})$$

Under the stochastic equicontinuity, the compactness of the parameter space, and the continuity of moment, we establish the uniform convergence (Newey and McFadden, 1994).

$$\sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{i=1}^n \{m(\mathbf{L}_{n,i}; \theta) - \mathbb{E}\{m(\mathbf{L}_{n,i}; \theta)\}\} \right| \xrightarrow{p} 0. \quad (\text{A.31})$$

Therefore, under the GMM regularity conditions described above,

$$\sup_{\theta \in \Theta} \left| \widehat{Q}_n(\theta) - Q_n(\theta) \right| \xrightarrow{p} 0. \quad (\text{A.32})$$



Finally, under the GMM regularity conditions described above, we have (i)  $Q_n(\theta)$  is uniquely minimized at  $\theta_0$ , (ii) parameter space  $\Theta$  is compact, (iii)  $Q_n(\theta)$  is continuous, and (iv) the uniform convergence (equation (A.32)). Therefore, Theorem 2.1 of Newey and McFadden (1994) implies

$$\hat{\theta} \xrightarrow{p} \theta_0,$$

which completes the proof of consistency.  $\square$

**Asymptotic Normality.** Next, we show asymptotic normality.

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} \text{Normal}(0, \Sigma)$$

where

$$\begin{aligned} \Sigma &= \Gamma_0 \text{Var} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n m(\mathbf{L}_{n,i}; \theta_0) \right) \Gamma_0^\top, \\ \Gamma_0 &= (M_0^\top \Omega M_0)^{-1} M_0^\top \Omega, \quad M_0 = \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left\{ \frac{\partial}{\partial \theta} m(\mathbf{L}_{n,i}; \theta_0) \right\}. \end{aligned}$$

**Proof:** By definition, we have

$$\hat{\theta} = \text{argmin}_{\theta \in \Theta} \hat{Q}_n(\theta)$$

We take the first order condition.

$$\frac{\partial \hat{Q}_n(\hat{\theta})}{\partial \theta} = 0$$

Using the mean-value expansion, we have

$$\begin{aligned} \sqrt{n}(\hat{\theta} - \theta_0) &= - \left\{ \frac{\partial^2 \hat{Q}_n(\tilde{\theta})}{\partial \theta \theta^\top} \right\}^{-1} \times \sqrt{n} \frac{\partial \hat{Q}_n(\theta_0)}{\partial \theta} \\ &= - \left\{ \frac{\partial^2 \hat{Q}_n(\tilde{\theta})}{\partial \theta \theta^\top} \right\}^{-1} \times \left\{ \frac{1}{n} \sum_{i=1}^n \frac{\partial}{\partial \theta^\top} m(\mathbf{L}_i; \theta_0) \right\}^\top \Omega \frac{1}{\sqrt{n}} \sum_{i=1}^n m(\mathbf{L}_i; \theta_0) \end{aligned}$$

where  $\tilde{\theta}$  is a mean value, located between  $\hat{\theta}$  and  $\theta_0$ , and

$$\begin{aligned} \left[ \frac{\partial^2 \hat{Q}_n(\tilde{\theta})}{\partial \theta \partial \theta^\top} \right]_{jk} &= \left\{ \frac{1}{n} \sum_{i=1}^n \frac{\partial}{\partial \theta_j} m(\mathbf{L}_i; \tilde{\theta}) \right\}^\top \Omega \left\{ \frac{1}{n} \sum_{i=1}^n \frac{\partial}{\partial \theta_k} m(\mathbf{L}_i; \tilde{\theta}) \right\} \\ &\quad + \left\{ \frac{1}{n} \sum_{i=1}^n \frac{\partial^2}{\partial \theta_j \partial \theta_k} m(\mathbf{L}_{n,i}; \tilde{\theta}) \right\}^\top \Omega \left\{ \frac{1}{n} \sum_{i=1}^n m(\mathbf{L}_{n,i}; \tilde{\theta}) \right\}. \end{aligned}$$

Therefore, under the GMM regularity conditions, Assumption 3, and consistency of  $\hat{\theta}$ ,

$$\begin{aligned} \left\{ \frac{\partial^2 \hat{Q}_n(\tilde{\theta})}{\partial \theta \partial \theta^\top} \right\}^{-1} &\xrightarrow{p} (M_0^\top \Omega M_0)^{-1}, \\ \left\{ \frac{1}{n} \sum_{i=1}^n \frac{\partial}{\partial \theta^\top} m(\mathbf{L}_i; \theta_0) \right\}^\top \Omega &\xrightarrow{p} M_0^\top \Omega. \end{aligned}$$

Thus,

$$\sqrt{n}(\hat{\theta} - \theta_0) = -(M_0^\top \Omega M_0)^{-1} M_0^\top \Omega \times \frac{1}{\sqrt{n}} \sum_{i=1}^n m(\mathbf{L}_{n,i}; \theta_0) + o_p(1).$$

Finally, under Assumption 3, the Cramér–Wold device and the network CLT (Theorem 3.2) by Kojevnikov *et al.* (2020) imply

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n m(\mathbf{L}_{n,i}; \theta_{0,n}) \xrightarrow{d} \mathcal{N} \left( 0, \text{Var} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n m(\mathbf{L}_{n,i}; \theta_0) \right) \right).$$

By combining the results using the Slutsky’s theorem, we obtain the desired result.

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} \text{Normal}(0, \Sigma),$$

which completes the proof.  $\square$

### B.3 Proof of Theorem 4

We consider asymptotic properties of the network HAC variance estimator. In addition to the regularity conditions required to prove Theorem 3, we also require the following regularity conditions for the choice of kernel and bandwidth. With  $p$  that satisfies Assumption 3,

$$\lim_{n \rightarrow \infty} \sum_{s \geq 0} |\omega(s/b_n) - 1| \rho_n(s) \beta_{n,s}^{1-2/p} = 0 \quad \text{a.s.},$$

where  $\rho_n(s)$  measures the average number of network peers at the distance  $s$ ,  $\rho_n(s) = \frac{1}{n} \sum_{i=1}^n \mathcal{N}_n(i; s)$ .

**Proof:** Given that  $\hat{\theta}$  is a consistent estimator of  $\theta_0$ , using the continuous mapping theorem under the GMM regularity condition, we need to prove that

$$\tilde{\Lambda}_n = \sum_{s \geq 0} \omega(s/b_n) \left\{ \frac{1}{n} \sum_{i \in \mathcal{N}_n} \sum_{j \in \mathcal{N}_n(i; s)} m(\mathbf{L}_{n,i}; \theta_0) m(\mathbf{L}_{n,j}; \theta_0)^\top \right\}.$$

is a consistent estimator of  $\Lambda_0$ . Because we assume that  $m(\mathbf{L}_{n,i}; \theta_0)$  is  $\psi$ -weakly dependent (Assumption 3), under the regularity condition on the choice of kernel and bandwidth (equation (18)), Proposition 4.1 of Kojevnikov *et al.* (2020) implies that  $\tilde{\Lambda}_n$  is a consistent estimator for  $\Lambda_0$ .

Moreover, under Assumption 3 and the GMM regularity conditions, we obtain consistency of  $\widehat{M}$ :  $\widehat{M} - M_0 \xrightarrow{p} 0$ , where  $\widehat{M} = \frac{1}{n} \sum_{i=1}^n \frac{\partial}{\partial \theta} m(\mathbf{L}_{n,i}; \hat{\theta})$  and  $M_0 = \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left\{ \frac{\partial}{\partial \theta} m(\mathbf{L}_{n,i}; \theta_0) \right\}$ . Finally, we can combine the results to obtain the desired result.

$$\widehat{\Sigma} - \Sigma \xrightarrow{p} 0$$

where

$$\begin{aligned} \Sigma &= \Gamma_0 \Lambda_0 \Gamma_0^\top, & \widehat{\Sigma} &= \widehat{\Gamma} \widehat{\Lambda} \widehat{\Gamma}^\top \\ \Gamma_0 &= (M_0^\top \Omega M_0)^{-1} M_0^\top \Omega, & \widehat{\Gamma} &= (\widehat{M}^\top \Omega \widehat{M})^{-1} \widehat{M}^\top \Omega, \end{aligned}$$

which completes the proof.  $\square$

## B.4 Proof of Lemma 4

Under Assumption 4.1, the ACPE can be represented as a linear function of parameters  $\gamma$  in the outcome confounding bridge function. Under this setting, it is sufficient to obtain multivariate asymptotic normality and consistent variance estimator for  $\gamma$ . As a result, we can simplify the moment function to be

$$\tilde{m}(\mathbf{L}_{n,i}; \gamma) = \{Y_{i2} - h(W_i, A_i, \mathbf{X}_i; \gamma)\} \times \eta(A_i, \mathbf{Z}_i, \mathbf{X}_i).$$

Under Assumption 4.2, there exists integer  $s^*$  such that for units  $i, j$  with the distance  $d_n(i, j) \geq s^*$ ,

$$\mathbf{L}_{n,j} \perp\!\!\!\perp \mathbf{L}_{n,i} \mid A_i, \mathbf{Z}_i, \mathbf{X}_i, U_i.$$

For such  $s^*$  and units  $i, j$ , we have

$$\tilde{m}(\mathbf{L}_{n,j}; \gamma_0) \perp\!\!\!\perp \tilde{m}(\mathbf{L}_{n,i}; \gamma_0) \mid A_i, \mathbf{Z}_i, \mathbf{X}_i, U_i. \quad (\text{A.33})$$

In addition, under Assumptions 2.2 and 2.3, we have

$$\mathbb{E}\{\tilde{m}(\mathbf{L}_{n,i}; \gamma_0) \mid A_i, \mathbf{Z}_i, \mathbf{X}_i, U_i\} = 0. \quad (\text{A.34})$$

Combining equations (A.33) and (A.34), we obtain

$$\begin{aligned} & \mathbb{E}\{\tilde{m}(\mathbf{L}_{n,i}; \gamma_0)\tilde{m}(\mathbf{L}_{n,j}; \gamma_0)^\top \mid A_i, \mathbf{Z}_i, \mathbf{X}_i, U_i\} = 0 \\ \implies & \mathbb{E}\{\tilde{m}(\mathbf{L}_{n,i}; \gamma_0)\tilde{m}(\mathbf{L}_{n,j}; \gamma_0)^\top\} = 0. \end{aligned}$$

for integer  $s^*$  and units  $i, j$  with the distance  $d_n(i, j) \geq s^*$ . Therefore,

$$\Lambda_0 = \sum_{s=0}^{s^*-1} \Lambda_0(s)$$

where

$$\Lambda_0(s) = \left\{ \frac{1}{n} \sum_{i \in N_n} \sum_{j \in \mathcal{N}_n(i; s)} \mathbb{E}\{\tilde{m}(\mathbf{L}_{n,i}; \gamma_0)\tilde{m}(\mathbf{L}_{n,j}; \gamma_0)^\top\} \right\}.$$

We can obtain its estimator as follows.

$$\hat{\Lambda}_{s^*} = \sum_{s=0}^{s^*-1} \omega(s/b_n) \left\{ \frac{1}{n} \sum_{i \in N_n} \sum_{j \in \mathcal{N}_n(i; s)} \tilde{m}(\mathbf{L}_{n,i}; \hat{\gamma})\tilde{m}(\mathbf{L}_{n,j}; \hat{\gamma})^\top \right\}.$$

Finally, we obtain the variance estimator for  $\hat{\gamma}$ .

$$\widehat{\text{Var}}(\hat{\gamma}) = \frac{1}{n} \hat{\Gamma}_\gamma \hat{\Lambda}_{s^*} \hat{\Gamma}_\gamma^\top. \quad (\text{A.35})$$

where  $\hat{\Gamma}_\gamma = (\widehat{M}_\gamma^\top \Omega \widehat{M}_\gamma)^{-1} \widehat{M}_\gamma^\top \Omega$ , and  $\widehat{M}_\gamma = \frac{1}{n} \sum_{i=1}^n \frac{\partial}{\partial \gamma} \tilde{m}(\mathbf{L}_{n,i}; \hat{\gamma})$ , which completes the proof.  $\square$

## B.5 Heterogeneous Expectation

In Section 3.5, we assume that the expectation of the causal peer effect,  $\mathbb{E}\{Y_{i2}(a) - Y_{i2}(a') \mid \mathcal{G}_n\}$ , is constant across units, while we allow for network-dependent (non-independent) errors. Here, to examine the heterogeneous expectation, we explicitly write out the conditioning on  $\mathcal{G}_n$ . In this section, we allow for heterogeneous expectation across units. As we observe only one sample of interconnected units in a single network, we have to make some assumptions to make progress. In this vein, we assume that  $\mathbb{E}\{Y_{i2}(a) - Y_{i2}(a') \mid \mathcal{G}_n\}$  depends only on a summary statistic of network  $\mathcal{G}_n$ , which we denote by vector  $\mathbf{g}_i$ . For example,  $\mathbf{g}_i$  could be the network-degree of unit  $i$ , centrality of unit  $i$ , or other network summary statistics. This is a common assumption scholars make in practice, and is similar to the idea of the exposure mapping (Aronow and Samii, 2017), which is used to reduce dimensionality of the potential outcomes.

Formally, we assume  $\mathbb{E}\{Y_{i2}(a) - Y_{i2}(a') \mid \mathcal{G}_n\} = \mathbb{E}\{Y_{i2}(a) - Y_{i2}(a') \mid \mathbf{g}_i\}$ . We then posit a model for the conditional expectation  $\mathbb{E}\{Y_{i2}(a) \mid \mathbf{g}_i\}$  with shared coefficients. This allows us to accommodate heterogeneous expectation across units in the network, while we can still make statistical inference about the target estimand with network-dependent errors.

As a concrete example, consider the following linear model with coefficients  $\varphi$ .

$$\mathbb{E}\{Y_{i2}(a) \mid \mathbf{g}_i\} = \varphi_0 + \varphi_1 \cdot a + \{\ell(\mathbf{g}_i)^\top \varphi_2\} \cdot a$$

where  $\ell(\mathbf{g}_i)$  is a user-specified function of  $\mathbf{g}_i$ . Under this model, we can re-write the ACPE as follows.

$$\tau(a, a') := \frac{1}{n} \sum_{i=1}^n \mathbb{E}\{Y_{i2}(a) - Y_{i2}(a') \mid \mathcal{G}_n\} = \varphi_1 \cdot (a - a') + (\bar{\ell}^\top \varphi_2) \cdot (a - a')$$

where  $\bar{\ell} = \frac{1}{n} \sum_{i=1}^n \ell(\mathbf{g}_i)$ . To estimate the ACPE, we first modify the moment function as follows.

$$m^\dagger(\mathbf{L}_{n,i}; \theta) = \left\{ \tau + (\ell(\mathbf{g}_i) - \bar{\ell})^\top \varphi_2 \cdot (a - a') \right\} - \{h(W_i, a, \mathbf{X}_i; \gamma) - h(W_i, a', \mathbf{X}_i; \gamma)\},$$

where  $\theta = (\tau, \varphi_2, \gamma)$ . We then show that  $\mathbb{E}\{m^\dagger(\mathbf{L}_{n,i}; \theta) \mid \mathcal{G}_n\} = 0$  for all  $i \in N_n$ . We start with the first term.

$$\begin{aligned} \mathbb{E} \left\{ \tau + (\ell(\mathbf{g}_i) - \bar{\ell})^\top \varphi_2 \cdot (a - a') \mid \mathcal{G}_n \right\} &= \varphi_1 \cdot (a - a') + \ell(\mathbf{g}_i)^\top \varphi_2 \cdot (a - a') \\ &= \mathbb{E}\{Y_{i2}(a) - Y_{i2}(a') \mid \mathcal{G}_n\}. \end{aligned}$$

We next consider the second term. Under Assumption 2,

$$\mathbb{E} \left( \{h(W_i, a, \mathbf{X}_i; \gamma) - h(W_i, a', \mathbf{X}_i; \gamma)\} \mid \mathcal{G}_n \right) = \mathbb{E}\{Y_{i2}(a) - Y_{i2}(a') \mid \mathcal{G}_n\},$$

which shows that  $\mathbb{E}\{m^\dagger(\mathbf{L}_{n,i}; \theta) \mid \mathcal{G}_n\} = 0$  for all  $i \in N_n$ . Therefore, we can use the following moment functions to estimate the ACPE  $\tau(a, a')$ .

$$m^*(\mathbf{L}_{n,i}; \theta) = \left\{ \begin{array}{l} m^\dagger(\mathbf{L}_{n,i}; \theta) \times \eta^*(\mathbf{g}_i) \\ \{Y_{i2} - h(W_i, A_i, \mathbf{X}_i; \gamma)\} \times \eta(A_i, \mathbf{Z}_i, \mathbf{X}_i) \end{array} \right\},$$

where  $\eta^*(\mathbf{g}_i) = (1, \ell(\mathbf{g}_i)^\top)^\top$ . Under the same assumption used in Section 3.5, we can consistently estimate the ACPE and construct an asymptotic confidence interval.  $\square$

## C Simulation Study under Violation of Assumptions

Here, we provide additional simulation studies to investigate the performance of the proposed DNC estimator in settings where some key identification assumptions are violated. In Section C.1, we consider violation of the negative control assumption (Assumption 2.2). In Section C.2, we consider violation of the outcome confounding bridge assumption (Assumption 2.3) due to violation of the completeness condition.

### C.1 Violation of Negative Control Assumptions

**Setup.** In this section, we consider violations of the negative control assumption (Assumption 2.2). In particular, we modify the data generating mechanism of Section 4 as follows. For units  $i = 1, \dots, n$ ,

- (1) Unobserved confounder with network dependence:  $U_i = \sum_{s \geq 0} \zeta^s \sum_{j \in \mathcal{N}(i; s)} \tilde{U}_j / |\mathcal{N}(i; s)|$  where  $\zeta = 0.8$  and  $\tilde{U}_j \stackrel{\text{i.i.d.}}{\sim} \text{Normal}(0, 1)$ . This part is the same as the one used in Section 4.
- (2) Observed covariates with network dependence:  $\mathbf{X}_i = (X_{i1}, X_{i2}, X_{i3})$  where, for  $k \in \{1, 2, 3\}$ ,  $X_{ik} = \sum_{s \geq 0} \zeta^s \sum_{j \in \mathcal{N}(i; s)} \tilde{X}_{jk} / |\mathcal{N}(i; s)|$ ,  $\zeta = 0.8$ , and  $\tilde{X}_{jk} \stackrel{\text{i.i.d.}}{\sim} \text{Normal}(0, 1)$ . This part is the same as the one used in Section 4.
- (3) Observed auxiliary variable:  $C_i = \sum_{s \geq 0} (\zeta_C)^s \sum_{j \in \mathcal{N}(i; s)} \tilde{C}_j / |\mathcal{N}(i; s)|$  where  $\tilde{C}_i = U_i + \beta_c^\top \mathbf{X}_i + \epsilon_{i0}$  where  $\epsilon_{i0} \stackrel{\text{i.i.d.}}{\sim} \text{Normal}(0, 1)$  and  $\beta_c = (0.05, 0.05, 0.05)$ . This part is the difference from the one used in Section 4.
- (4) Focal behavior at the baseline:  $Y_{i1} = U_i + 0.05C_i + \beta_1^\top \mathbf{X}_i + \epsilon_{i1}$  where  $\epsilon_{i1} \stackrel{\text{i.i.d.}}{\sim} \text{Normal}(0, 1)$  and  $\beta_1 = (-1, -1, -1)$ . This part is the same as the one used in Section 4.
- (5) Focal behavior at the follow-up:  $Y_{i2} = \tau A_i + 0.2Y_{i1} + 3U_i + 0.05C_i + \beta_2^\top \mathbf{X}_i + \epsilon_{i2}$  where  $\epsilon_{i2} \stackrel{\text{i.i.d.}}{\sim} \text{Normal}(0, 1)$ , and  $\beta_2 = (-1, -1, -1)$ . The treatment variable  $A_i$  is defined as  $A_i = \sum_{j \in \mathcal{N}(i; 1)} Y_{j1} / |\mathcal{N}(i; 1)|$ . This part is the same as the one used in Section 4.

The main and only difference is in (3) where we allow for network association between auxiliary variable  $C$  across units. Because we use  $W_i = C_i$  as NCO, and  $Z_i = \sum_{j \in \mathcal{N}(i; 1)} C_j / |\mathcal{N}(i; 1)|$  as NCE, this network association violates assumptions for NCO and NCE (Assumption 2.2).

We consider three different levels of the violation using parameter  $\zeta_C \in \{0.02, 0.10, 0.50\}$ . We call them “Small”, “Moderate”, and “Large” violations in Table A1. We fix sample size to be 1000, and we generate 2000 simulations to evaluate estimators in terms of the absolute mean bias, the standard error (computed as the standard deviation of point estimates across simulations), the root mean squared error (RMSE), and coverage of 95% confidence intervals based on the network HAC variance estimator. We standardize the first three quantities by the true ACPE to ease interpretation.

**Results.** Table A1 summarizes the results of the simulation study. Our proposed DNC estimator has small bias and has reasonable coverage when the violation is “small.” However, as we expect, the larger is the violation, the bias is larger and coverage performance becomes poorer.

Simulation Design		DNC				
Network	Violation	Bias	Standard Error	RMSE	Coverage (Analytical)	Coverage (Default)
SW-4	Small	0.04	0.38	0.39	0.94	0.93
	Moderate	0.32	0.30	0.45	0.78	0.78
	Large	0.74	0.25	0.78	0.12	0.12
SW-8	Small	0.02	0.53	0.53	0.95	0.94
	Moderate	0.26	0.43	0.50	0.87	0.86
	Large	0.66	0.35	0.74	0.46	0.46
Add Health	Small	0.03	0.41	0.41	0.94	0.93
	Moderate	0.34	0.34	0.48	0.80	0.78
	Large	0.81	0.28	0.86	0.13	0.13

Table A1: Operating Characteristics when the Negative Control Assumptions are Violated.

*Note:* We consider three different levels of violation: “Small” ( $\zeta_C = 0.02$ ), “Moderate” ( $\zeta_C = 0.10$ ), and “Large” ( $\zeta_C = 0.50$ ). We examine the same three different networks; the small world network model with the average degree of four (SW-4) and eight (SW-8), and the Add Health network. For the DNC estimator, we report the absolute mean bias, the standard error, the RMSE, and coverage of the 95% confidence intervals based on the analytical bandwidth and the default bandwidth. The absolute mean bias, the standard error, and the RMSE for both estimators are standardized by the true ACPE.

## C.2 Violation of Confounding Bridge Assumption due to Completeness

**Setup.** In this section, we consider violations of the outcome confounding bridge assumption (Assumption 2.3). In particular, we consider violation of the completeness condition (Assumption 5) we use to prove the existence of an outcome confounding bridge function.

In particular, we modify the data generating mechanism of Section 4 as follows. For units  $i = 1, \dots, n$ ,

- (1) Two unobserved confounders with network dependence: For  $k \in \{1, 2\}$ ,  $U_{ik} = \sum_{s \geq 0} \zeta^s \sum_{j \in \mathcal{N}(i; s)} \tilde{U}_{jk} / |\mathcal{N}(i; s)|$  where  $\zeta = 0.8$  and  $\tilde{U}_{jk} \stackrel{\text{i.i.d.}}{\sim} \text{Normal}(0, 1)$ . This part is the difference from the one used in Section 4.
- (2) Observed covariates with network dependence:  $\mathbf{X}_i = (X_{i1}, X_{i2}, X_{i3})$  where, for  $k \in \{1, 2, 3\}$ ,  $X_{ik} = \sum_{s \geq 0} \zeta^s \sum_{j \in \mathcal{N}(i; s)} \tilde{X}_{jk} / |\mathcal{N}(i; s)|$ ,  $\zeta = 0.8$ , and  $\tilde{X}_{jk} \stackrel{\text{i.i.d.}}{\sim} \text{Normal}(0, 1)$ . This part is the same as the one used in Section 4.
- (3) Observed auxiliary variable:  $C_i = U_{i1} + \beta_{UC} U_{i2} + \beta_c^\top \mathbf{X}_i + \epsilon_{i0}$  where  $\epsilon_{i0} \stackrel{\text{i.i.d.}}{\sim} \text{Normal}(0, 1)$  and  $\beta_c = (0.05, 0.05, 0.05)$ . The part of  $U_{i2}$  is the difference from the one used in Section 4.
- (4) Focal behavior at the baseline:  $Y_{i1} = U_{i1} + \beta_{UY1} U_{i2} + 0.05 C_i + \beta_1^\top \mathbf{X}_i + \epsilon_{i1}$  where  $\epsilon_{i1} \stackrel{\text{i.i.d.}}{\sim} \text{Normal}(0, 1)$  and  $\beta_1 = (-1, -1, -1)$ . The part of  $U_{i2}$  is the difference from the one used in Section 4.
- (5) Focal behavior at the follow-up:  $Y_{i2} = \tau A_i + 0.2 Y_{i1} + 3 U_{i1} + \beta_{UY2} U_{i2} + 0.05 C_i + \beta_2^\top \mathbf{X}_i + \epsilon_{i2}$  where  $\epsilon_{i2} \stackrel{\text{i.i.d.}}{\sim} \text{Normal}(0, 1)$ , and  $\beta_2 = (-1, -1, -1)$ . The treatment variable  $A_i$  is defined as  $A_i = \sum_{j \in \mathcal{N}(i; 1)} Y_{j1} / |\mathcal{N}(i; 1)|$ . The part of  $U_{i2}$  is the difference from the one used in Section 4.

The main difference is in (1) where we allow for two separate unmeasured confounders  $U_{i1}$  and  $U_{i2}$ . Yet, we use  $W_i = C_i$  as NCO, and  $Z_i = \sum_{j \in \mathcal{N}(i; 1)} C_j / |\mathcal{N}(i; 1)|$  as NCE. Therefore, the number of unmeasured confounders is larger than the number of NCO, and this violates the completeness condition (Assumption 5). In this case, an outcome confounding bridge does not exist and Assumption 2.3 is violated.

We consider three different levels of violation using parameters  $(\beta_{UC}, \beta_{UY1}, \beta_{UY2})$ . We define “Small”, “Moderate”, and “Large” violations as follows.

- “Small”:  $\beta_{UC} = 0.1, \beta_{UY1} = \beta_{UY2} = 0.005$
- “Moderate”:  $\beta_{UC} = 0.25, \beta_{UY1} = \beta_{UY2} = 0.0125$
- “Large”:  $\beta_{UC} = 0.5, \beta_{UY1} = \beta_{UY2} = 0.025$

We fix sample size to be 1000, and we generate 2000 simulations to evaluate estimators in terms of the absolute mean bias, the standard error (computed as the standard deviation of point estimates across simulations), the root mean squared error (RMSE), and coverage of 95% confidence intervals based on the network HAC variance estimator. We standardize the first three quantities by the true ACPE to ease interpretation.

**Results.** Table A2 summarizes the results of the simulation study. Our proposed DNC estimator has small bias and has reasonable coverage when the violation is “small.” However, as we expect, the larger is the violation, the bias is larger and coverage performance becomes poorer.

Simulation Design		DNC				
Network	Violation	Bias	Standard Error	RMSE	Coverage (Analytical)	Coverage (Default)
SW-4	Small	0.02	0.41	0.41	0.94	0.94
	Moderate	0.20	0.38	0.43	0.89	0.89
	Large	0.75	0.38	0.84	0.39	0.39
SW-8	Small	0.07	0.56	0.56	0.95	0.95
	Moderate	0.11	0.54	0.56	0.91	0.91
	Large	0.58	0.48	0.75	0.68	0.68
Add Health	Small	0.04	0.44	0.45	0.95	0.94
	Moderate	0.17	0.44	0.47	0.89	0.89
	Large	0.70	0.41	0.81	0.49	0.49

Table A2: Operating Characteristics when the Confounding Bridge Assumption and the Completeness Condition are Violated.

*Note:* We consider three different levels of violation: “Small”, “Moderate”, and “Large” (see above for their definitions). We examine the same three different networks; the small world network model with the average degree of four (SW-4) and eight (SW-8), and the Add Health network. For the DNC estimator, we report the absolute mean bias, the standard error, the RMSE, and coverage of the 95% confidence intervals based on the analytical bandwidth and the default bandwidth. The absolute mean bias, the standard error, and the RMSE for both estimators are standardized by the true ACPE.